THE MATHEMATICAL GAZETTE

EDITED BY
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VOL. XXIV. DECEMBER, 1940. No. 262

THE RESEARCH METHOD IN TEACHING MATHEMATICS.

BY E. A. G. KNOWLES.

IN the discussion held at the annual meeting of the Mathematical Association in January 1936* on "Rider-Work in Geometry", various interesting proposals were made and examples put forward to show how school geometry might be given a research aspect by means of the "discovery rider" or the "inductive method" in order to develop the full powers of the mind and to stimulate the creative abilities. Teachers who have tried to put into practice the ideas presented on that occasion will have obtained rather varied, if not actually contradictory, results. However many may have been successful there are always quite a number who come to the false conclusion that the average pupil has no creative abilities and that all efforts in that direction will fail, since only the very few mathematical specialists are found by them to benefit by it.

The reasons for such failures are that the highest powers of the mind, namely, the creative ones, are the most sensitive, and unless special care is taken, they remain latent or are even destroyed. The harmful effects which undermine the work of the pupils can be avoided by using a special method, the "research method", which was discovered by M. Vaerting.† After years of experiments she published her results in a small book. In the second and third editions a practical part has been added, giving actual examples of the working of the method in a first course in geometry, thus proving that the method is applicable from the earliest stages.

This method can be applied not only in schools but also in universities, in fact one might perhaps go so far as to say that it is the only method truly worthy of a university. At any rate it would be

* See *Math. Gazette*, Vol. XX., No. 238.

† *Neue Wege im mathematischen Unterricht.* Berlin 1921. 2nd Ed. 1929. 3rd Ed. 1932.

a great pity if the good work that had been started in schools were undone in the universities, and if, in consequence, students leaving the universities to become teachers were less fitted for their profession than when they left school.

In the following paragraphs it is endeavoured to give a rough outline of the research method and its educational value as far as that can be done within a small space.

1. *The aim of the research method and the main means for its realisation.*

The aim of the research method in teaching mathematics is to make it possible for students to acquire their mathematical knowledge by their own activity, by independent, creative thought, *i.e.* by their own research. The word "research" is commonly used only for work which leads to results hitherto unknown in our civilisation, whereas the word as used here means work which involves methods and leads to results hitherto unknown to the pupils or students concerned. This use of the word may be justified by the fact that the nature of the work is the same, whether its results be new to the world or not, provided only that they are new to the person engaged in the work.

Apart from the fact that no textbooks are used (though for students the use of a library with good books of reference, but no textbooks, is valuable), the two chief features of the research method are the following: (*a*) The solution in writing of problems in the true sense of the word, *i.e.* questions or tasks demanding new thoughts from the pupils or students beyond those already known. These are the source of their mathematical knowledge.

It is important that problems should be attempted first in *written* work, not in general discussions, in order that quiet concentration may be possible for each individual. Only in this way can all the different faculties of the intelligence be aroused and protected from interference, and only thus can the maximum effort be produced.

(*b*) The use of the students' own notebooks, containing the formulae and theorems discovered by themselves and arranged and expressed according to their own taste and judgment (though under guidance of the teacher) as an aid to the memory in all their work, including examinations. The free use of notebooks or collections of formulae is necessary in order to avoid putting undue strain on the memory and so to leave the mind free for participation in the production of new ideas. Everyone engaged in research naturally makes the freest use of his own records. The abolition of all intentional memorising work is imperative if the method is to be successful. The investigations of M. Vaerting* have shown that intentional memorising has a destructive influence on the creative faculties and slowly transforms original and highly gifted personalities into merely receptive and industrious automata. In sensitive individuals the destructive influence can go so far that even their receptiveness suffers. Such influences can often be observed in first-year students

* *Die Vernichtung der Intelligenz durch Gedächtnisarbeit*, 1913.

who have just survived examinations ; some take months to recover from the harm they have suffered. The quality of their work sometimes seems entirely to disagree with their good school reports, and their recovery at a later stage alone proves that their teachers' judgment was not entirely wrong ; and it is impossible to guess how many really gifted students never recover, or have had their originality destroyed before anyone has had a chance to notice it.

The destructive effects of memorising are immediate, and all experiences with the research method show that the whole success of the method is undermined by them. It is therefore desirable that the use of collections of formulae should be allowed in all mathematical examinations.

2. *The effects of true problems compared with examples.*

Usually the examples worked by students do not develop their creative ability, the main purpose of such examples being to test the understanding and to practise methods in order to fix knowledge imparted to them by their teachers or textbooks. Although a certain resourcefulness and skill in manipulation (not possessed by all people alike, but not to be identified with a real mathematical ability) is quite often necessary in solving some of the harder examples, yet on the whole no original ideas are demanded, mere reproduction usually being sufficient. The greater the ability of the student, the more this is true. Those most capable of original work get only the smallest chance of developing their abilities for it. This explains a fact I have personally observed among students, namely, that some of the apparently least gifted students at times develop a surprising originality of thought. One's appreciation of such originality is often disturbed by the length, apparent clumsiness and unsuccessfulness of the work, but trying to regard the work without prejudice or impatience one cannot but admire the idea in it and wonder why the more gifted students show no such signs of originality. The explanation is this : for the more gifted students the question is a mere example of methods already known, and they are therefore able to do it without calling into play their capacity for original thought ; on the other hand, for the less gifted students, who have gaps in their understanding, the question is a true problem which arouses their interest and consequently their higher creative powers.

This explanation is illustrated by another fact. I have noticed that the best students do not always produce the best or most thoroughly thought-out solution to *every* problem. Sometimes a student of apparently medium ability produces the best thought-out solution. Again, the problem was too easy for the best but just reached the limit of efficiency of a medium student, who, through interest and effort, produced the best solution. From this it can be expected that, if set at the right moment, before the ideas have been anticipated, the example might prove to be a real problem, even for the more gifted students. This thought was verified on

later occasions, where I put questions in advance of the program of the lectures and thus was calling for original thought from *all* students. This was particularly so in the case where the question set demanded too much from the majority but was just within the limit of efficiency of the best members of the class, say four or five out of thirty, it appeared at times that the number of different methods used was about equal to the number of solutions, and then also sometimes traces of originality could be observed in the work of the more gifted students and revealed indications as to the nature of their talent.

A particularly instructive case was that of a problem on series of complex numbers, given before such series had been dealt with in lectures. It happened to be an especially suitable problem from the point of view of the research method, and owing to the geometrical interpretation of complex numbers it gave scope to geometrical as well as algebraical or merely manipulative talents. In this case four students found a complete solution, three of which were entirely different from each other. Only a few of the students had made no effort at all. As I usually let students who have found a good solution demonstrate it to the class, in this case I let all three solutions be produced and this was received with interest by the class in general.

The appearance of original thought together with a variety in methods of solution is a criterion which enables the teacher to tell whether the work given to the students was a true problem for them or only an example. This is an index for him of his success or failure in trying to realise the research method. It is the most important and at the same time the most difficult task of the teacher to be able to distinguish independent, creative thought from reproductive, merely imitative thought. It requires all his instinct and ability not only as a teacher but also as a mathematician.

3. *How the claims of the individual and the majority are satisfied.*

It is a characteristic feature of the research method that the teacher does not try to set only questions that could, or rather should, be solved by the majority. This is definitely harmful for the development of the abilities of the more gifted ones, and although the more gifted ones should, for reasons of justice, not be favoured more than the rest, yet they have a right to demand *equal* opportunity. Through the research method *all* pupils and *all* students, the most gifted as well as the least gifted ones, are given opportunity to test and to practise their creative abilities on true problems. The teacher therefore should not regard it as a deplorable accident if he sets a problem that is only solved by one or two, in fact he must see that such problems occur just as often as those solved by the majority or the whole class, and in trying to achieve this it cannot be avoided that occasionally a problem proves too hard to be solved by anyone. This, as Vaerting points out, is good for the more gifted ones, because it saves them from developing conceit,

and it consoles the less gifted ones if they are not always the only ones to fail.

Never should the teacher, either in the wording of the problem or in any explanation or demonstration, give a hint or impart knowledge that could be found by any students, even if it were only by one, or else he has not utilised the independent and creative abilities of the class to the utmost. Very often the hardest problem has to be given first and then easier ones to follow. The latter are very necessary in order to prevent discouragement.

It would, especially in large classes, mean carrying the principle too far, if one expected *every* member of the class to discover all the required mathematical knowledge entirely by his (or her) own research. This would take too long, but it should be true of the class as a whole.

It might be said as an argument against the research method that the setting of problems which test the efficiency, independence and originality of the more gifted members of the class to the utmost, must be too heavy a burden and therefore harmful for the less gifted ones. The practice of the method has proved that this is not the case. Its result is the awakening of a real interest in mathematics in the whole class, and it succeeds in raising the efficiency of *all* its members.

The above example on complex numbers showed these effects clearly. By their unsuccessful endeavours the whole class is acquainted with the problem and appreciates its difficulties, and those who have tried in vain look forward to its solution, their interest having been roused, and their abilities for understanding thereby having been increased. This was actually proved by the fact that, as far as I could see, the whole class listened to all three solutions, whereas I had allowed them the freedom to carry on any work of their own instead, if they preferred to, and many of them usually make use of this freedom.

4. *Combination of research method and lecture method.*

Although I cannot claim to have been able to realise the research method to any degree satisfactory to me yet, owing to very many obstacles, some of which, for reasons explained before, are due to the examination system and also to the lecture system which educates the students to an attitude of passivity, yet the results of experiments like the above have encouraged me to think that in time it would be possible to put the method completely into practice both in universities and technical colleges, much to the benefit of the students.

Obviously the complete realisation of the research method would mean abolition of all lectures. Whether this is desirable or an ideal sort of combination of lectures with research method is preferable, only future experience and carefully conducted experiments can reveal.

On the whole, without any exaggeration, I can safely say that I have found my experiments successful in increasing the inde-

pendence and self-responsibility of the students, and in all cases of failure I was able to detect a disturbing cause, *other* than insufficient ability or lack of goodwill on part of the students. On the contrary, I was often surprised and pleased at the great amount of goodwill with which the students responded to my experiments.

In their second-year course I gave the students a share in my lectures. Under my guidance they carried on the revision themselves, to which half of the lectures were devoted. It was done in this way: I gave two of them a set of questions which, whenever possible, demanded a certain amount of original thought from them, and tried to throw a new light on the subject already known. These they worked out, coming to me to discuss difficulties, after they had tried their best, and then they produced their work in the class, having divided it between them as they pleased. Most of them seemed to do this work with real interest, and some took quite an amount of trouble to consult books and quite noticeably they also had the desire to be able to answer all questions put to them by the class. Quite often they were more thorough and did more than I had asked of them. In one case in particular a simple remark of mine about the generality of a result caused the student to go into the subject much more deeply and to produce a derivation of Bernoulli-numbers which surprised and highly interested the whole class in spite of the student's remark that they need not remember it for their examination!

They really felt responsible for the success of that part of the lecture entrusted to them. Some students, who seemed shy and less successful at first in talking to the class, developed surprisingly. Through interest in the subject self-consciousness was lost, and quite a number of really good little lectures were produced. One student developed abilities to such an extent as to contradict bad reports that I heard about him in other subjects.

I found that if the little lectures produced by the students did not last much longer than ten minutes each, the result was best, and the class did not get too tired of listening to a less successful effort; time also was left for discussion, and if this proved unnecessary the ordinary course of lecture was resumed.

Although my efforts then were very imperfect and many improvements still required, on which I have since been working, yet I think they give rise to the hope of a successful realisation of the research method in undergraduate courses. Perhaps giving the students a share in lectures may lead to an ideal combination of research method and lecture method.

5. *The research method in teaching young children.*

Certainly the success of the research method in teaching mathematics at universities would be very much greater if all students had been used to that method at school. Again, for school many reasons can be found that seem to lead to the conclusion that those (if any) failures in first attempts at realising a creative method in

the first lessons in mathematics at school are caused by the preceding teaching of arithmetic by methods quite opposed to the research method.

My own experiences with younger children from five to seven years of age have brought me to the conviction that the research method in the correct sense of the word may well be applied with great success in their first attempt at gaining knowledge and may even be a very great help in pleasantly acquiring the most tiresome technique of reading and writing. The first steps in arithmetic can be taught as a true branch of mathematics, as a research on the laws of number. Writing exercises become to a high degree superfluous if, even after the first few months, the child begins to keep a notebook in which it stores up the knowledge of the discoveries it has made. It should be allowed to use its own simple and vivid language, illustrated by drawings. This notebook becomes a treasure to the child and it delights in writing well and arranging the matter nicely. Some children who ordinarily are a burden to the teacher on account of their bad writing, suddenly begin to write much better, because their work is inspired by a real interest and purpose, being not merely a monotonous exercise.

When knowing only addition, a child invented multiplication, and I have seen the face of a child light up with joy at its discovery of the commutative law of multiplication, which at later stages is only a dry and dull thing to learn, because too much experience has made it too obvious to be of interest. Having once directed the natural spirit of discovery, born in every child, into the region of the science of number, spontaneously the child produces questions and problems, some not very hard, others much more so, some too hard, giving it automatically all the practice in manipulation that it needs besides continual stimulation to further work and further discoveries. Having discovered a law the joy is so great that the child is happy and in fact desires to make further examples " to see the law work " again and again and to invent means of its use as a tool. This is the most natural method of " practising " ; it is just like the child's (or even the adult's) attitude towards a new toy.

If therefore, as is often found, modern teaching which tends to encourage the creative abilities of the child, does not succeed in giving the child the necessary skill in manipulation, this is due to some removable cause given by the method of teaching.

Using the research method one can again and again convince oneself that the psychology of the child changes vastly, and quite often the result of researches on the ability for logical thought and deductive reasoning are completely contradicted. It has usually been found that the desire and ability for logical deduction does not exist in the younger child, but only develops much later, and for this reason it is considered that an experimental stage in teaching mathematics should precede the deductive stage.* In this way the

* Compare *The Teaching of Geometry in Schools*, a report prepared for the Mathematical Association, London, 1923.

child is to be slowly brought to recognise the necessity for a rigorous proof. Using the research method one finds that he (the child) does not generalise results carelessly, and in fact demands and seeks for proofs more often than the teacher expects. The rigour and clarity of the child's reasoning are in fact very striking, but in order to be able to appreciate this, the teacher needs to have true mathematical abilities as well as an understanding of the child's psychology.

6. *General educational value of the research method.*

The value of the research method in teaching mathematics is great, if only because it is the surest method of detecting mathematical and technical ability. But from the point of view of general education, the teaching of mathematics must provide more than a test for a special ability if its existence as a compulsory school subject is to be justified. It must have an educational value for every student, not only the mathematical specialist.

A number of good effects which mathematical teaching is supposed to have on the pupil are quite well known, so, for instance, the development of abilities for logical reasoning, and also the development of some ethical qualities like objectivity and love of truth have been emphasised. Nevertheless, investigating these effects more closely,* it has been found that they are to a very high degree, if not entirely, dependent on the method of teaching.

The highest form of mathematical work is research. This requires all the abilities that the occupation with mathematics can at all call forth. In studying the effects of mathematical research on the person engaged in research we therefore study the effects of mathematical work in its pure form, not affected by the disturbing influences of any particular method of teaching. The research method therefore makes it possible to study the effects of true mathematical work on all students.

The fact that it is possible to realise the research method in teaching mathematics is due to a peculiarity of mathematics as a science, which distinguishes it from all other branches of knowledge. This is, that all mathematical thought is based on a few fundamental assumptions which are obvious or can be made obvious to every child, while the objects of mathematical thought are creations of our mind of which everyone is capable. Nothing beyond this has to be taken for granted ; nothing has to be believed without a possibility of conviction. But only if the research method is used and the mathematical knowledge grows naturally in the child's own mind, by its own effort, can we be sure that full conviction is always reached. That is why in true mathematical work, as made possible by the research method, the child can get experience entirely on its own and thus gain the power and freedom of its own thought at an early age. This raises courage and self-confidence and inspires further activity and further success.

* See E. Knowles : *Die Forschungsmethode im mathematischen Unterricht als Mittel der Erziehung zu Autonomie und Gemeinschaft.* Berlin, 1933.

It can be shown that the more an activity calls forth *all* the abilities and powers of the mind, the more the work approaches research work, the more does the experience of a success in mathematical work become an experience of success in mental activity in general; and thereby it raises courage and self-confidence in general. These lead to further successful activity and belong to the most important foundations for the formation of character, which mathematics more than any other subject can give at an early age.

The lack of courage caused by the old methods of teaching can also be proved by the fact that when the students have *not* been taught by a creative method they prefer questions with a given answer. They need the support of an authority, whereas students taught by the research method have a real desire to discover everything for themselves.

This example of the difference in the effect of two methods of teaching illustrates a fact the proof of which could not be attempted in a short space. When taught by a bad method, mathematics may destroy the student's confidence in his own power of logical reasoning, and increase blind belief in " authorities " and ruin originality and independence of judgment. The mind may be so mechanised that ideas will run automatically along certain grooves, no matter whether they are appropriate to the subject or not. A good example of this is given in the above-mentioned discussion on " Rider Work in Geometry ". We can only be thankful that the transfer of formal abilities, or in this case rather disabilities, is limited. Yet if such methods dominate all subjects and the entire education, then the danger is very great that they will affect the general attitude of the character and produce characters who always aim at conforming to general prejudice without courage to attempt any thought of their own, nor allowing others to do so, in short what H. G. Wells calls the " herd intelligence " that " tyrannises here and yields to tyranny there ".* The research method encourages characters with a love for truth, who are able to put aside prejudice, in order not to be tempted to go the wrong way, but to have the courage to make the best efforts within their ability to find the right way.

Only the attitude developed by true mathematical work can ultimately render successful the valuable educational work of H. G. Wells and others in imparting correct information; for the desire to impart must be met by an ability to discard wrong and imperfect information and an ever alive desire to take in new information.

Correct information *alone* cannot transform a herd into a community of human beings, but the character of the individual is decisive. Masses of knowledge and best experiences do not suffice, but we need the ability to *use* them and to interpret them correctly, and for this courage and independence of thought and utmost objectivity are ever necessary.

* In his address on " The Informative Content of Education " at the British Association, Nottingham, 1937.

The information of to-day is the legend of to-morrow. Fact is but a fleeting moment in the passage of time. Information is ever imperfect, and each new day makes new demands. A lively mind with a spirit of discovery, ready for new information and having ability for independent thought to deal with it, this alone is equal to the demands of the day; this alone can transform the herd into a community.

Within this space it is not possible to prove all the valuable educational influences in the interest of the individual as well as the community which it is possible to develop by the research method in teaching mathematics. This would lend more support to the above statements, but I must refer to the investigation mentioned before.

For H. G. Wells mathematics is only a manipulation with symbols, and therefore he does not group it among the subjects of informative value. This seems to me to prove that he has not been taught mathematics by the research method, for then he would know that mathematics yields as much knowledge about life and movement and structure in our actual world as history does, only within another sphere. For if we grant that the objects of mathematics are only symbols, creations of our own mind, which here and there we apply to actual space or actual circumstances and which we can alter and remodel according to requirements of the actual world, must we not also say that the objects of history—Greece, Rome, Napoleon, etc.—are only symbols of collections of facts long gone by and inaccessible to us? Are they not also, in spite of the best and most earnest endeavours of historians, creations of our own mind occasionally remodelled according to the political needs of the day? Therefore granting mathematics no informative value, we might be equally justified in denying it to history.

Mathematics, as taught by the research method, grows in the mind of the child from experiments in the actual world, from thought and imagination too. It is inspired again and again in the mind of the child by all that it sees and can make in the actual world, as was the case in the historical development of mathematics in the minds of mathematicians of past ages. Thus mathematics, having arisen from the actual world, is bound to give us information about it. One of the aims of modern geometry is to describe the laws that govern motion, to make us able to form an image in our minds of the behaviour of the bodies in the world we live in. Thus it represents a part of the information we can get from the world we live in.

Perhaps the hope is not quite unjustified that, if mathematics is taught by the research method, it will be possible to raise it above the level of a mere technique or manipulation of symbols, at best serving as a kind of mental athletics, or a source for riddles and puzzles to kill time by "light amusements", but to give it a place among the subjects of true culture and education.

E. A. G. K.

CYCLICALLY SYMMETRIC KNIGHT'S TOURS.

By W. H. Cozens.

Several mathematicians have studied the question of symmetry in the knight's tour, one of the earliest being Euler and one of the latest Dr. Kraitchik of Brussels. Their investigations have been in the nature of random experiments, and they have confined themselves chiefly to square boards, thus missing such elegant results as

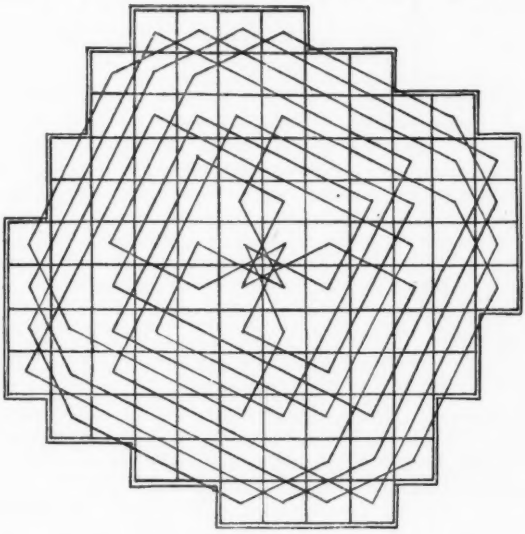


Fig. 1.

Fig. 1, which was published anonymously in the German periodical *Denken und Raten* in 1931.

The following is an attempt to treat the subject systematically, and establish a method by which it would be possible to enumerate every possible cyclically symmetric tour on every possible board. The following definitions will be necessary :

The knight's move. The knight, that " irresponsible low comedian of the chessboard ", moves along the diagonal of a rectangle 3 squares by 2. On an open board it has 8 moves, forming what is known as the knight-wheel (Fig. 2). A fundamental consideration is that on a chequered board every move is from a black square to a white one, or *vice versa* ; consequently, to get from any square to another of the same colour, a knight must take an even number of moves, and to get to a square of the other colour it must take an odd number.

A knight's tour. A series of consecutive knight's moves, visiting every square of some given board once without visiting any one twice and *finishing at the starting point* is a knight's tour. Some writers loosely speak of any perambulation of the knight as a tour,

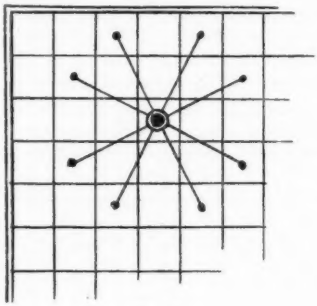


FIG. 2.

but the word " tour " strictly implies a return to the starting-point. Unfinished or open tracks had better be spoken of as " knight-routes " or " knight-tracks ". A tour may be recorded either geometrically (Fig. 1) or arithmetically (Fig. 3); those who are sufficiently interested will probably enjoy tracing out an arithmetically-recorded tour on squared paper.

Cyclic symmetry. Geometrically the test is that the tour shall be unaltered by a rotation of 90° (Fig. 1). Arithmetically it is that the numbers in homologous squares shall differ by one-fourth of the total number of squares toured. In Fig. 3 the difference is $8/4 = 2$.

1	4	7
6		2
3	8	5

FIG. 3.

Since the board must itself be cyclically symmetric it is convenient to regard it as being formed by the removal of certain parts of a square board, which I will call the original square. The following conventions seem reasonable. Squares may be removed from the edge of the original square or from its centre, or both, but not elsewhere. The resultant board must be in one piece : parts in contact only at a corner of a square and not joined by at least one edge are not admissible. There are three cases according as the side of the original square is :

A : not divisible by 2 ;
B : divisible by 2 but not by 4 ;
C : divisible by 4.

Case A. The most obvious consideration here is that we have an odd number of squares, and therefore an extra square of one colour, so that on the full board no cyclically symmetric tour—indeed no re-entrant tour at all—is possible. It is therefore always essential to remove the middle square. Moreover, since homologous squares will be of the same colour, each quarter of the board must consist of an even number of squares ; and since the square of any odd number is one more than a multiple of 4, the removal of the central square is not only necessary but sufficient to make a board susceptible to cyclic symmetry. (This does not mean that such a tour will always be possible, for other factors, to be considered presently, may intervene.) To reduce the size of the board, or to alter its shape, it is clear that an even number of squares must be taken from each quarter. The rule is, therefore :

> Remove $8k+1$ squares, k being zero or any positive integer. The odd square must be the central one, and the others must be half of each colour, disposed in cyclic symmetry around the centre.

Case B. Here the board is symmetrical, not around a square, but around a point, so that homologous squares are of different colours, and each quarter of the tour must contain an odd number of moves ; and since the side of the original square is not divisible by 4 it is clear that the complete board fulfils the condition in this case. (Note : the smallest square board which may be toured in cyclic symmetry is therefore 6^2. The five possible results are shown in Figs. 12–16.) An even number of squares may be removed from each corner to produce other boards. The rule is :

> Remove $8k$ squares, k being as before.

Case C. As in Case B the quarter board must contain an odd number of squares, but this time an odd number will evidently have to be taken from each corner. Rule :

> Remove $8k+4$ squares.

In Case B and Case C it is of course necessary that the squares removed should be symmetrically disposed.

Starting with any original square the procedure is first to remove, successively, all combinations of the appropriate numbers of squares, first taking $k=0$, then 1, then 2, and so on as far as the size of the original square allows. Each of the resulting boards is then examined to ascertain how many—if any—cyclically symmetric tours are possible upon it. The method is best explained by an example.

Take the board shown in Fig. 4, which has been formed by the removal of nine squares from the 5^2 original square. Consider one quarter of the board (say the part enclosed by the heavy line in

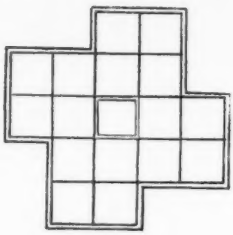


FIG. 4.

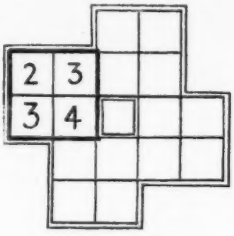


FIG. 5.

Fig. 5). Make a note, on each square, of how many moves would be open to a knight standing on that square. Should any square have no communications at all, it is an island, and of course renders any tour out of the question. If a square has only one route of communication, the tour is likewise impossible, for if a knight entered this square it would then have no exit. If a square is bi-communicant, as in our example, it is evident that the knight has but one route through that square : the different sense will not alter the geometrical aspect of the tour. The moves to and from bi-communicant squares may therefore be drawn (Fig. 6). These lines

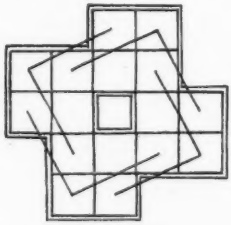


FIG. 6.

are compulsory and form part of every possible tour on this board. Two " loose ends " are now to be found in the quarter marked out in Fig. 5. Against each of these mark how many *vacant* squares are now within a knight's move. In both cases it is two. Select either end—say the left-hand one—and join it to either of its vacant squares—say the upper one—and repeat this in the other three quarters (Fig. 7). Now draw a second figure, repeat Fig. 6, and join the same end as before to its other alternative, the lower square, and repeat cyclically (Fig. 8).

In Figs. 7 and 8 we now have every square visited ; it remains to examine whether in either case, or in both, the four tracks may be united into one cyclically symmetric tour. Call *any* loose end in Fig. 7 A1. Trace the track, and label the other end A2. Proceeding clockwise, say, call the square corresponding to A1, B1, and

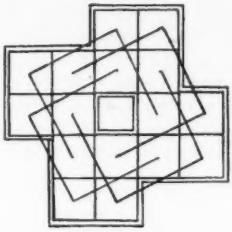


FIG. 7.

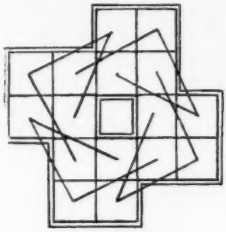


FIG. 8.

trace it to B2 ; similarly with C1, C2 and D1, D2. Now if A1 is a knight's move from B2, or if A2 is a knight's move from B1, they may be joined and a cyclically symmetric tour results (Fig. 9). If the ends are joined other than as specified above the result may be :

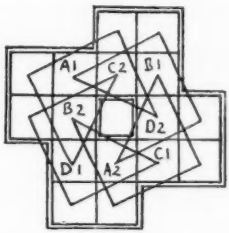


FIG. 9.

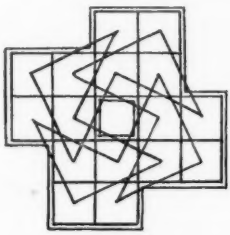


FIG. 10.

(i) 4 separate closed tours ;
(ii) 2 separate closed tours ;
(iii) one tour, but only diagonally symmetric, that is, unaltered by a rotation of 180°, but altered by a rotation of 90°.

Fig. 8, similarly treated, gives another result (Fig. 10). These must be the only two tours possible on this particular board.

This is an exceedingly simple and straightforward case. It would not be possible in a paper such as the present to deal in detail with a larger board, for although the method is the same the complications may become formidable. In practice it is absolutely essential to use some regular system of numbering the variations and sub-variations. Supposing, for instance, that there is no bi-communicant square on a certain board, we start with a tri-communicant one. The three knight-moves combine in pairs to give three different routes through the square. Call these A, B and C. Proceeding with A, we find that the loose ends give, say, 2 and 4 continuations respectively. Take the two, of course, and label these possibilities A1 and A2. Then continuing with A1 we may subdivide it into A1A, A1B and A1C, and so on, using letters and figures alternately so that the progenitors of any diagram under consideration are clear

from its reference number. By the time we complete a tour—say, A1A2A1B—we shall have a whole series of diagrams, in varying stages of incompleteness, to which we must return. Many, of course, prove to be blind alleys. The more proficient one becomes, the more time one saves by recognising, several moves ahead, that some tour will become impossible. Care is sometimes necessary to avoid producing tours which are not new but only reflections of previously discovered ones.

Finally, here is a table of results :

Original square 3^2 (Case A) :

$k=0$; 1 board ; 1 tour (Fig. 3).

Original square 4^2 (Case C) :

No cyclically symmetric tour is possible.

Original square 5^2 (Case A) :

$k=0$; no result.
$k=1$; 2 boards ; 3 tours in all (Figs. 9, 10, 11).

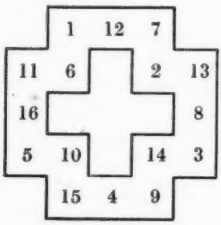


FIG. 11.

Original square 6^2 (Case B) :

$k=0$; 1 board ; 5 tours (Figs. 12–16).

1	14	35	6	3	28
12	7	2	27	34	5
15	36	13	4	29	26
8	11	22	31	18	33
23	16	9	20	25	30
10	21	24	17	32	19

FIG. 12.

1	8	31	16	3	10
30	15	2	9	24	17
7	36	23	32	11	4
22	29	14	5	18	25
35	6	27	20	33	12
28	21	34	13	26	19

FIG. 13.

1	16	31	24	3	10
30	23	2	9	32	25
15	36	17	26	11	4
22	29	8	35	18	33
7	14	27	20	5	12
28	21	6	13	34	19

FIG. 14.

1	26	5	16	3	28
12	15	2	27	6	17
25	36	13	4	29	32
14	11	22	31	18	7
35	24	9	20	33	30
10	21	34	23	8	19

FIG. 15.

1	26	13	16	3	28
12	15	2	27	6	17
25	36	23	14	29	4
22	11	32	5	18	7
35	24	9	20	33	30
10	21	34	31	8	19

FIG. 16.

This last is a correction of Kraitchik,* who gives the first four of this set as the only possible solutions :

$k = 1$; 3 boards ; 8 results in all (Figs. 17–24).

1		27		3	8
24	19	2	7	26	
	28	25	4	9	6
20	23	18	11	14	
	12	21	16	5	10
22	17		13		15

FIG. 17.

1		27		3	22
10	5	2	21	26	
	28	11	4	23	20
6	9	18	25	14	
	12	7	16	19	24
8	17		13		15

FIG. 18.

* *Le Problème du Cavalier*, 1927.

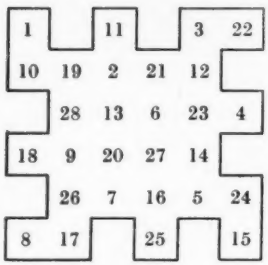


FIG. 19.

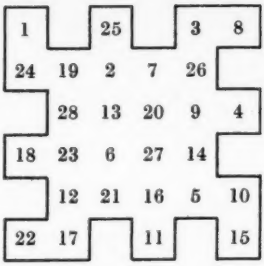


FIG. 20.

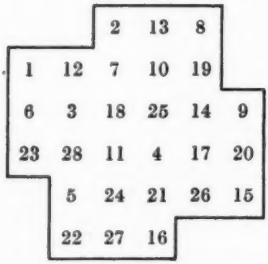


FIG. 21.

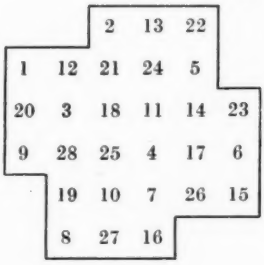


FIG. 22.

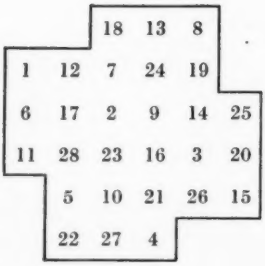


FIG. 23.

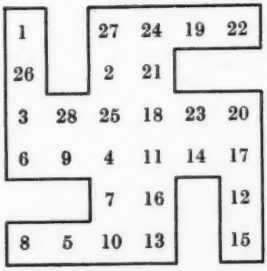


FIG. 24.

$k=2$; 3 boards ; 4 results in all (Figs. 25–28).

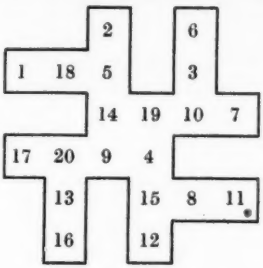


FIG. 25.

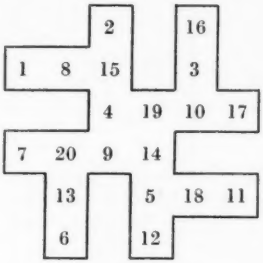


FIG. 26.

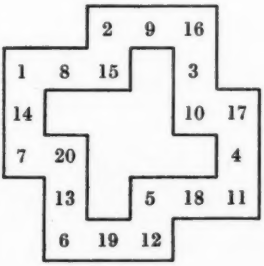


FIG. 27.

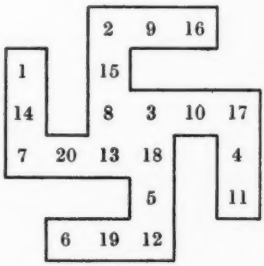


FIG. 28.

Original square 7^2 (Case A) :

$k=0$;	1 board ;	64 tours.
$k=1$;	3 boards ;	88 tours in all.
$k=2$;	5 boards ;	45 tours in all.
$k=3$;	2 boards ;	10 tours in all.

The diagrams of these tours are not given.

These 228 tours exhaust all boards up to 7^2. Beyond this I have not made any systematic study, but the 8^2 board (the familiar chess and draughts board) would undoubtedly yield over a thousand results.

W. H. COZENS.

GLEANINGS FAR AND NEAR

1331. When they heard of the Restoration of 1660, it was said that a mathematician died of joy and a translator of Rabelais died laughing. The conduct of both was subsequently justified.—John Dickson Carr, *The Murder of Sir Edmund Godfrey* (1936), p. 19.

THE CIRCLE IN HOMOGENEOUS COORDINATES.

By N. M. Gibbins.

1. It is necessary to challenge Mr. F. H. V. Gulasekharam's Note 1448 (May, 1940) in which he claims to give simpler methods than usual for dealing with the circle

$$S \equiv (\Sigma lx)(\Sigma x) - \Sigma a^2yz = 0.$$

Objection must be taken at the very beginning where he says "Suppose the circle cuts the sides of the triangle of reference ABC at D_1, D_2". The arguments of §§ 1 and 2 are sound when these points are real, but it is asking too much of a young student to expect him to understand what is meant by the ratios D_1C/BD_1 and D_2C/BD_2 when D_1 and D_2 are imaginary. Moreover the writer considers that the arguments of §§ 3 and 4 are not simple but very difficult.

There is no need at all for reference to imaginary or infinite elements. The whole matter can be put on the secure foundation of a definition and a proposition in pure geometry, which every young student ought to know or can be quickly taught. The definition is that of the radical axis of two circles, namely, that it is the line perpendicular to their line of centres through the point thereon such that the difference of the squares of the segments is equal to the difference of the squares of the radii. The proposition is that the difference of the powers of a point with respect to the two circles is equal to twice the product of the length of the perpendicular from the point to their radical axis by the distance between their centres.

2. To implement this plan we need an expression for the power of a point P with respect to the circumcircle, centre O, of the triangle of reference. This may be found by pure geometry as in *Gazette*, XXIII, p. 347; or as follows. The power required, defined as $OP^2 - R^2$, must be a quadratic function of the trilinear, and therefore of the areal, coordinates of P. Let it be equal to

$$\lambda x^2 + \mu y^2 + \nu z^2 + \lambda' yz + \mu' zx + \nu' xy.$$

Then the power of $A(1, 0, 0)$ is zero and so $\lambda = 0$. Similarly $\mu = \nu = 0$. Also the power of the mid-point of BC, that is, $(0, \frac{1}{2}, \frac{1}{2})$, is $-\frac{1}{4}a^2$. Hence $\lambda = -a^2$ and similarly $\mu = -b^2$, $\nu = -c^2$. Hence the required expression is $-\Sigma a^2yz$, where x, y, z are actual areal coordinates, or $-(\Sigma a^2yz)/(x+y+z)^2$, when x, y, z are proportional thereto.

Now if p, q, r are the lengths of the perpendiculars from A, B, C on a line, we know that the length of the perpendicular from P on it is $(px+qy+rz)/(x+y+z)$. If then this line is the radical axis of a circle centre K and the circle ABC, the difference of the powers of P with respect to the two circles is

$$2 \,.\, OK \,.\, (px+qy+rz)/(x+y+z).$$

Also the difference (in the same order) of the powers of A with respect to the circles is $2p \,.\, OK$. But the power of A with respect

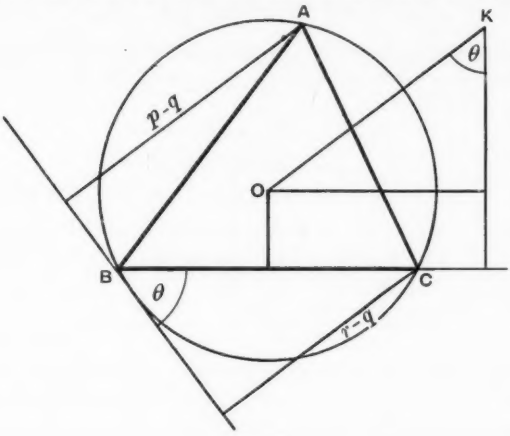


FIG. 1.

to the circle ABC is zero. Thus the power of A with respect to the circle centre K is $2p \,.\, OK = \varpi_A$, say; and similarly for B and C. Hence the power of P with respect to the circle K is

$$(\varpi_A x + \varpi_B y + \varpi_C z)/(x+y+z) - (\Sigma a^2 yz)/(x+y+z)^2$$
$$= \{(\varpi_A x + \varpi_B y + \varpi_C z)(x+y+z) - \Sigma a^2 yz\}/(x+y+z)^2.$$

Thus the equation of a circle in areal coordinates is of the form

$$S \equiv (lx + my + nz)(x+y+z) - a^2yz - b^2zx - c^2xy = 0,$$

and when actual areal coordinates are used, so that $x+y+z \equiv 1$, l, m, n are the powers of A, B, C with respect to the circle, and the power of any other point (x_1, y_1, z_1) is S_1.

3. Let (x_1, y_1, z_1) be the centre of the circle K and (x_0, y_0, z_0) that of ABC.

Then $$2\Delta(x_1 \sim x_0) = a \,.\, OK \cos\theta,$$

where θ is the angle the radical axis makes with BC.

Draw the parallel through B to the radical axis.

Then $$\sin\theta = (r-q)/a \quad \text{and} \quad \sin(\theta + B) = (p-q)/c.$$

Hence $$(r-q)\cos B/a + \cos\theta \,.\, \sin B = (p-q)/c\,;$$

that is, $$ca \sin B \cos\theta = a(p-q) + c\cos B \,.\, (q-r)$$

or $$2\Delta \cos\theta = a \,.\, p - b\cos C \,.\, q - c\cos B \,.\, r.$$

Hence $$2OK \,.\, 2\Delta \cos\theta = al - b\cos C \,.\, m - c\cos B \,.\, n$$
$$= -\frac{1}{2a}\frac{\partial\Omega}{\partial l},$$

where $$\Omega = \Sigma a^2 l^2 - \Sigma 2bcmn \cos A.$$

Hence $$\frac{a}{2\Delta} \cdot OK \cos\theta = -\frac{1}{16\Delta^2} \cdot \frac{\partial\Omega}{\partial l}.$$

To determine the sign of $x_1 - x_0$, take the circle on CA as diameter; then $l=0$, $m=ca\cos B$, $n=0$, $x_1=\frac{1}{2}$; *vide* § 5, Ex. (*c*), XXIII, p. 345.

Then
$$\begin{aligned}\partial\Omega/\partial l &= -2ab\cos C \,.\, ca\cos B \\ &= -8\Delta \,.\, Ra\cos B\cos C \\ &= -8\Delta \,.\, Ra\,.\,(-\cos A + \sin B\sin C) \\ &= +8\Delta \,.\, Ra\cos A - 8\Delta^2\end{aligned}$$

and
$$\begin{aligned}-\frac{1}{16\Delta^2}\frac{\partial\Omega}{\partial l} &= -(Ra\cos A)/2\Delta + \tfrac{1}{2} \\ &= -x_0 + \tfrac{1}{2}.\end{aligned}$$

Hence $$x_1 = x_0 - \frac{1}{16\Delta^2}\frac{\partial\Omega}{\partial l},$$

and similarly $$y_1 = y_0 - \frac{1}{16\Delta^2}\frac{\partial\Omega}{\partial m},$$

$$z_1 = z_0 - \frac{1}{16\Delta^2}\frac{\partial\Omega}{\partial n}.$$

Hence also
$$\begin{aligned}l(x_1-x_0) + m(y_1-y_0) + n(z_1-z_0) &= -\frac{1}{16\Delta^2}\Sigma l\frac{\partial\Omega}{\partial l} \\ &= -\Omega/8\Delta^2.\end{aligned}$$

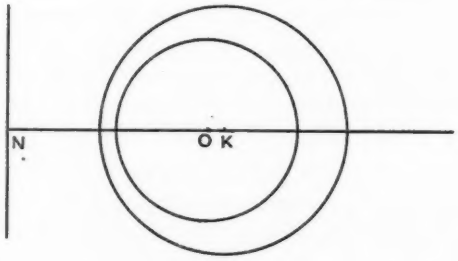


FIG. 2.

4. Let ρ be the radius of the circle K. Then by the definition
$$\begin{aligned}\rho^2 \sim R^2 &= KN^2 - ON^2 \\ &= KO(KN + ON) \\ &= KO(px_1 + qy_1 + rz_1 + px_0 + qy_0 + rz_0) \\ &= \tfrac{1}{2}(lx_1 + my_1 + nz_1 + lx_0 + my_0 + nz_0)\end{aligned}$$

$$= \tfrac{1}{2}\{l(x_1 - x_0) + m(y_1 - y_0) + n(z_1 - z_0)\} + lx_0 + my_0 + nz_0$$

$$= -\frac{\Omega}{16\Delta^2} + \frac{R}{2\Delta} \cdot \Sigma al \cos A.$$

Taking the same particular case as before, the right-hand side

$$= -\frac{b^2 \cdot c^2 a^2 \cos^2 B}{16\Delta^2} + \frac{R}{2\Delta} \cdot b \cdot ca \cos B \cos B$$

$$= (a^2 b^2 c^2 \cos^2 B)/16\Delta^2$$

$$= R^2 \cos^2 B$$

$$= R^2 - \tfrac{1}{4}b^2.$$

Hence, above, the right-hand side is equal to $R^2 - \rho^2$

and
$$\rho^2 = R^2 - \frac{R}{2\Delta} \Sigma al \cos A + \frac{\Omega}{16\Delta^2}.$$

5. *Examples.*

(*a*) With respect to the point-circle at A, the power of A is zero, the power of B is c^2, and that of C is b^2.

Hence $P_1A^2 = (c^2 y_1 + b^2 z_1)(x_1 + y_1 + z_1) - a^2 y_1 z_1 - b^2 z_1 x_1 - c^2 x_1 y_1$

$$= c^2 y_1^2 + b^2 z_1^2 + (b^2 + c^2 - a^2) y_1 z_1.$$

(*b*) The power of $P_2(x_2, y_2, z_2)$ with respect to the point-circle at P_1 is

$$(x_2 \cdot AP_1^2 + y_2 \cdot BP_1^2 + z_2 \cdot CP_1^2)(x_2 + y_2 + z_2) - \Sigma a^2 y_2 z_2$$

$$= x_2\{c^2 y_1^2 + b^2 z_1^2 + (b^2 + c^2 - a^2) y_1 z_1\}$$

$$+ y_2\{a^2 z_1^2 + c^2 x_1^2 + (c^2 + a^2 - b^2) z_1 x_1\}$$

$$+ z_2\{b^2 x_1^2 + a^2 y_1^2 + (a^2 + b^2 - c^2) x_1 y_1\}$$

$$- \Sigma a^2 y_2 z_2.$$

In this the coefficient of a^2 is

$$-x_2 y_1 z_1 + y_2 z_1 (z_1 + x_1) + z_2 y_1 (y_1 + x_1) - y_2 z_2$$

$$= -x_2 y_1 z_1 + y_2 z_1 (1 - y_1) + z_2 y_1 (1 - z_1) - y_2 z_2$$

$$= -y_1 z_1 + y_2 z_1 + z_2 y_1 - y_2 z_2$$

$$= -(y_1 - y_2)(z_1 - z_2).$$

Similarly for the coefficients of b^2 and c^2.

Hence
$$P_2P_1^2 = -\Sigma a^2 (y_1 - y_2)(z_1 - z_2).$$

(*c*) With respect to the circle on BC as diameter, the powers of B and C are zero, and that of A is $\frac{1}{2}(b^2 + c^2 - a^2)$ or $bc \cos A$. Hence the power of a point $P(x, y, z)$ with respect to this circle is

$$bc \cos A \cdot x(x + y + z) - \Sigma a^2 yz.$$

(*d*) The power of A with respect to the circle on P_1, P_2 as diameter is $\frac{1}{2}(AP_1^2 + AP_2^2 - P_1P_2^2)$ and, by Ex. (*b*), this is equal to

$$\begin{aligned}&-a^2y_1z_1 - b^2z_1(x_1 - 1) - c^2(x_1 - 1)y_1\\&-a^2y_2z_2 - b^2z_2(x_2 - 1) - c^2(x_2 - 1)y_2\\&+a^2(y_1 - y_2)(z_1 - z_2) + b^2(z_1 - z_2)(x_1 - x_2) + c^2(x_1 - x_2)(y_1 - y_2).\end{aligned}$$

In this the coefficient of a^2 is $-(y_1z_2 + y_2z_1)$.
The coefficient of b^2 is

$$\begin{aligned}&-(z_1x_2 + z_2x_1) + z_1 + z_2\\&= -(z_1x_2 + z_2x_1) + z_1(x_2 + y_2 + z_2) + z_2(x_1 + y_1 + z_1)\\&= (y_1z_2 + y_2z_1) + 2z_1z_2.\end{aligned}$$

Similarly the coefficient of c^2 is $(y_1z_2 + y_2z_1) + 2y_1y_2$.
Hence

$$\tfrac{1}{2}(AP_1^2 + AP_2^2 - P_1P_2^2) = c^2y_1y_2 + b^2z_1z_2 + bc\cos A\,(y_1z_2 + y_2z_1).$$

Similarly for the powers of B and C, and the power of any other point with respect to the circle may be written down at once.

COROLLARY. Calling the powers of A, B, C respectively l, m, n, we may calculate $al - b\cos C\,.\,m - c\cos B\,.\,n$ and rapidly deduce that

$$\partial\Omega/\partial l = 16\Delta^2\{x_0 - \tfrac{1}{2}(x_1 + x_2)\},$$

thus verifying the result of § 3. N. M. G.

BUREAU FOR THE SOLUTION OF PROBLEMS.

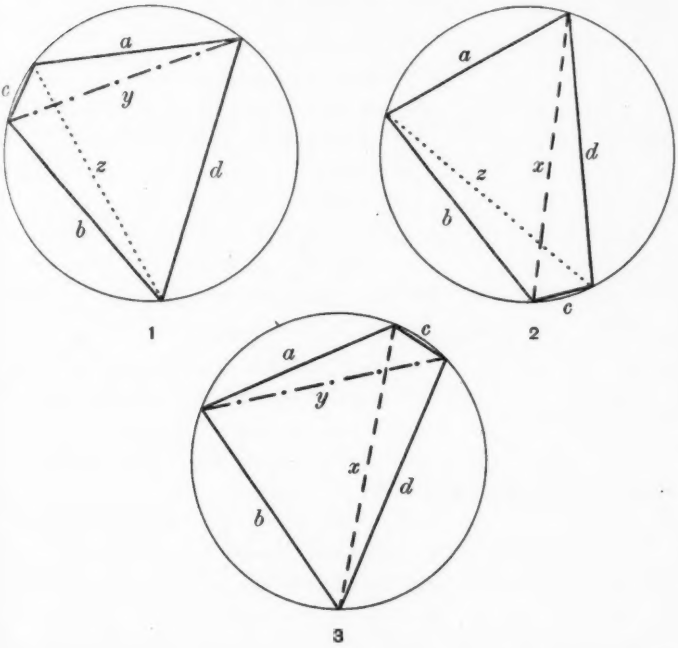
THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

1332. Hevelius lost his devoted wife on the 10 or 11 March, 1662. Although now more than fifty-one years of age, he took a second wife in the following year—the pretty sixteen-year old Catherina Elisabetha Koopman, daughter of another prosperous Danzig merchant.

"And when in the star-lit night she followed with enraptured gaze and beating heart, through his giant telescope, the shining full moon, on her silent path, she exclaimed with enthusiasm: 'To remain and gaze here always, to be allowed to explore and proclaim with you the wonder of the heavens; that would make me perfectly happy'! And the worthy man felt that it might make him happy too."—E. F. MacPike, *Hevelius, Flamsteed and Halley*, London, 1937, pp. 3 and 4. [Per Mr. Frank Robbins.]

RATIONAL CYCLIC QUADRILATERALS.

By A. P. Rollett.



I. *Diagonals rational.*

From four given lengths as sides it is possible in general to construct three distinct cyclic quadrilaterals, whose circumcircles are equal. Any pair of these quadrilaterals has one diagonal-length in common. Consequently, if a, b, c, d are the sides and x, y, z the diagonals, we have by Ptolemy's theorem

$$ab + cd = yz, \quad \ldots\ldots(1)$$

$$ac + db = zx, \quad \ldots\ldots(2)$$

$$ad + bc = xy. \quad \ldots\ldots(3)$$

Thus $(ab + cd) : (ac + db) : (ad + bc)$

$$= (1/x) : (1/y) : (1/z).$$

Hence if a, b, c, d, x, y are rational, z is also rational.

From (2) and (3), adding and factorising :

$$\left.\begin{aligned} (a+b)(c+d) &= x(y+z), \\ (a+c)(d+b) &= y(z+x), \\ (a+d)(b+c) &= z(x+y). \end{aligned}\right\} \quad \text{similarly and} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Let

$$\left.\begin{aligned} \lambda(a+b) &= y+z, \\ \mu(a+c) &= z+x, \\ \nu(a+d) &= x+y, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and

$$\left.\begin{aligned} c+d &= \lambda x, \\ d+b &= \mu y, \\ b+c &= \nu z. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Then

$$\begin{aligned} \lambda\mu\nu(a+b) &= \mu\nu(y+z) \\ &= \nu(\mu y)+\mu(\nu z) \\ &= \nu(d+b)+\mu(b+c). \end{aligned}$$

Hence $\lambda\mu\nu a+(\lambda\mu\nu-\mu-\nu)b-\mu c-\mu d=0.$

Let $\omega=\lambda+\mu+\nu-\lambda\mu\nu.$

Then

$$\begin{aligned} \lambda\mu\nu a &= (\omega-\lambda)b+ \quad \mu c+ \quad \nu d \\ &= \lambda b+(\omega-\mu)c+ \quad \nu d \\ &= \lambda b+ \quad \mu c+(\omega-\nu)d, \end{aligned}$$

and solving these equations we get

$$\left.\begin{aligned} a &= k(4-\omega^2), \\ b &= k(2\mu-\omega)(2\nu-\omega), \\ c &= k(2\nu-\omega)(2\lambda-\omega), \\ d &= k(2\lambda-\omega)(2\mu-\omega). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

There are, of course, restrictions on the values of the parameters λ, μ, ν. From the geometry of the figures, we have $c+d>x$, and $a+b>x$. From (6) we have $c+d=\lambda x$ and from (5) $\lambda(a+b)=y+z$. From these we deduce that $\lambda>1$, and that $y+z>x$. Similarly $\mu>1$, $\nu>1$, and $z+x>y$, $x+y>z$. Consequently x, y, z could be the sides of a triangle.

If $\lambda=1+\alpha$, $\mu=1+\beta$, $\nu=1+\gamma$, where α, β, γ are positive, we have

$$\omega=2-(\alpha\beta\gamma+\Sigma\beta\gamma),$$

so that $\omega<2$. Hence $\omega<2\lambda$, etc., which implies that b, c, d are positive if k is positive.

We must choose $\omega>-2$ if a is to be positive.

First, then, we choose $\lambda>1$; then μ such that

$$1<\mu<(\lambda+3)/(\lambda-1),$$

to ensure that $\nu>1$; lastly we choose ν such that

$$1<\nu<(\lambda+\mu+2)/(\lambda\mu-1)$$

to ensure that $\omega > -2$. The diagonals, obtained from (6), are

$$\left.\begin{aligned} x &= (c+d)/\lambda \\ &= 2k(2\lambda-\omega)(\mu+\nu-\omega)/\lambda \\ &= 2k(2\lambda-\omega)(\mu\nu-1), \\ y &= 2k(2\mu-\omega)(\nu\lambda-1), \\ z &= 2k(2\nu-\omega)(\lambda\mu-1). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

A simple particular case is obtained by taking $k=\frac{1}{4}$ and $\omega=0$. We then have $\lambda+\mu+\nu=\lambda\mu\nu$, $a=1$, $b=\mu\nu$, $c=\nu\lambda$, $d=\lambda\mu$, and

$$\left.\begin{aligned} x &= \lambda(\mu\nu-1) \\ &= \mu+\nu, \\ y &= \nu+\lambda, \\ z &= \lambda+\mu, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

with necessary restrictions on the values of λ, μ, ν apart from

$$\lambda+\mu+\nu=\lambda\mu\nu.$$

II. *Area and diagonals rational.*

Writing $2s=\Sigma a$ and $s_1=s-a$, etc., the area is

$$S=\sqrt{(s_1s_2s_3s_4)}.$$

We find $$s-a=2(\mu\nu-1)(\nu\lambda-1)(\lambda\mu-1),$$

$$s-b=2(\lambda^2-1)(\mu\nu-1), \text{ etc.},$$

and consequently

$$S=4(\mu\nu-1)(\nu\lambda-1)(\lambda\mu-1)\sqrt{\{(\lambda^2-1)(\mu^2-1)(\nu^2-1)\}}.$$

The rationalisation of S can be effected in several ways; for example:

(i) by taking λ, μ, ν of the form $(m^2+n^2)/(m^2-n^2)$;

(ii) by first choosing μ and ν, and then λ such that

$$(\lambda+1)(\mu-1)(\nu-1)=(\lambda-1)(\mu+1)(\nu+1).$$

But there is a much easier method for cyclic quadrilaterals with rational area and diagonals.

For, consider the cyclic quadrilaterals formed from s_1, s_2, s_3, s_4 as sides. We have

$$\begin{aligned} s_1s_2+s_3s_4 &= (s-a)(s-b)+(s-c)(s-d) \\ &= 2s^2-s\,.\,\Sigma a+ab+cd \\ &= ab+cd. \end{aligned}$$

Similarly, $$s_1s_3+s_2s_4=ac+bd,$$

and $$s_1s_4+s_2s_3=ad+bc.$$

Hence the diagonals of the new quadrilaterals will be of the same lengths as the diagonals x, y, z formed from the original sides a, b, c, d. Now

$$\Sigma s_1=4s-\Sigma a=2s,$$

so that the area of the derived quadrilateral is

$$\sqrt{\{(s-s_1)(s-s_2)(s-s_3)(s-s_4)\}} = \sqrt{(abcd)}.$$

If therefore a, b, c, d are formed in accordance with equations (7), this area will be

$$(2\lambda-\omega)(2\mu-\omega)(2\nu-\omega)\sqrt{(4-\omega^2)}$$

and will be rational if $\omega = 4mn/(m^2+n^2)$.

Any cyclic quadrilateral formed by taking $\omega = 4mn/(m^2+n^2)$ will thus give rise to another whose diagonals are the same but whose area is rational. In particular, if $\omega = 0$, giving the system (9), the area of the derived quadrilateral will be

$$S = \lambda\mu\nu = \lambda + \mu + \nu.$$

If the original quadrilateral is also circumscribable, having $a+c=b+d$, then the derived quadrilateral has the same sides, for

$$a+c=b+d=s,$$

giving $$s_1=c, \quad s_2=d, \quad s_3=a, \quad s_4=b.$$

III. *Area rational.*

If we choose any four rational numbers, s_1, s_2, s_3, s_4 whose product is a square, then the sides of a cyclic quadrilateral with rational area are

$$a=s-s_1, \quad b=s-s_2, \quad c=s-s_3, \quad d=s-s_4,$$

where $2s = \Sigma s_1$.

IV. *Area rational and the quadrilateral also circumscribable.*

If the sides are a, b, c, d in order, then $a+c=b+d$ and the area is $\sqrt{(abcd)}$. We can first choose four rational numbers α, β, γ, δ whose product is a square. Then if we choose p, so that

$$p\alpha+\gamma = p\beta+\delta,$$

then $a=p\alpha$, $b=p\beta$, $c=\gamma$, $d=\delta$, will, within limits, satisfy the required conditions ; or we may choose p so that

$$p\alpha+p\gamma = \beta+\delta,$$

and take $a=p\alpha$, $b=\beta$, $c=p\gamma$, $d=\delta$.

V. *Diagonals rational and quadrilateral circumscribable.*

If $a+c=b+d$, then from (5) and (6)

$$z+x=\mu^2 y.$$

This, on substitution and rearrangement, gives

$$\mu^2(\nu^2\lambda^2-1)-2\mu(\nu+\lambda)(\nu\lambda-1)+\nu^2+\lambda^2-2=0,$$

which has rational solutions if Δ is a square, where

$$\begin{aligned}\Delta &= (\nu+\lambda)^2(\nu\lambda-1)^2-(\nu^2\lambda^2-1)(\nu^2+\lambda^2-2)\\ &= 2(\nu\lambda-1)(\nu^2-1)(\lambda^2-1).\end{aligned}$$

We can make Δ a square by first choosing λ so that λ^2-1 is a square,

and then finding ν so that $2(\nu\lambda - 1)(\nu^2 - 1)$ is a square ; or we may choose $\lambda = (m^2 + 2n^2)/(m^2 - 2n^2)$ to make $2(\lambda^2 - 1)$ a square, and then take $\nu = (\lambda^2 + 4)/4\lambda$ to make $(\nu\lambda - 1)(\nu^2 - 1)$ a square. The usual conditions must also be satisfied.

VI. *Diagonals and area rational, and quadrilateral also circumscribable.*

There seems to be no solution.

Some examples.

I. Diagonals rational.

$\lambda = 1\frac{1}{6}$, $\mu = 1\frac{1}{3}$, $\nu = 1\frac{1}{2}$; $\omega = 1\frac{2}{3}$;
$a = 11$, $b = 12$, $c = 8$, $d = 6$; $x = 12$, $y = 13\frac{1}{2}$, $z = 13\frac{1}{8}$.
$\lambda = 2\frac{1}{2}$, $\mu = 1\frac{1}{2}$, $\nu = 1\frac{1}{11}$; $\omega = 1$;
$a = 33$, $b = 26$, $c = 52$, $d = 88$; $x = 56$, $y = 76$, $z = 71\frac{1}{2}$.
$\lambda = 2\frac{1}{3}$, $\mu = 1\frac{1}{3}$, $\nu = 1\frac{1}{2}$; $\omega = 0$;
$a = 72$, $b = 36$, $c = 63$, $d = 56$; $x = 48$, $y = 69$, $z = 56$.
$\lambda = 2$, $\mu = 1\frac{3}{4}$, $\nu = 1\frac{1}{2}$; $\omega = 0$;
$a = 4$, $b = 10\frac{1}{2}$ $c = 12$, $d = 14$; $x = 13$, $y = 14$, $z = 15$.

II. Diagonals and area rational.

$\lambda = 3$, $\mu = 1\frac{1}{2}$, $\nu = 1\frac{2}{9}$; $\omega = \frac{2}{9}$;
$a = 16$, $b = 25$, $c = 52$, $d = 65$; $x = 39$, $y = 60$, $z = 63$; $S = 1134$.
$\lambda = 2$, $\mu = 1\frac{3}{4}$, $\nu = 1\frac{1}{2}$; $\omega = 0$;
$a = 16$, $b = 42$, $c = 48$, $d = 56$; $x = 52$, $y = 56$, $z = 60$.

And the " derived " quadrilateral is (65, 39, 33, 25) with the same diagonals but with rational area 1344.

Similarly from

$\lambda = 2$, $\mu = 1\frac{2}{3}$, $\nu = 1\frac{4}{7}$; $\omega = 0$;
$a = 21$, $b = 55$, $c = 66$, $d = 70$; $x = 68$, $y = 75$, $z = 77$

we derive the quadrilateral (85, 51, 40, 36) with area 2310.

III. Area rational.

$s_1 = 2$, $s_2 = 3$, $s_3 = 4$, $s_4 = 6$ gives $a = 11$, $b = 9$, $c = 7$, $d = 3$; $S = 48$.

IV. Area rational and quadrilateral circumscribable.

$\alpha = 2$, $\beta = 4$, $\gamma = 6$, $\delta = 3$; $a = 7$, $b = 16$, $c = 21$, $d = 12$; $S = 168$.
$\alpha = 2$, $\beta = 6$, $\gamma = 4$, $\delta = 3$; $a = 1$, $b = 3$, $c = 8$, $d = 6$; $S = 12$.

V. Diagonals rational and quadrilateral circumscribable.

$\lambda = 1\frac{1}{4}$, $\mu = 1\frac{5}{14}$, $\nu = 2$; $\omega = 1\frac{3}{14}$;
$a = 55$, $b = 91$, $c = 78$, $d = 42$; $x = 96$, $y = 98$, $z = 84\frac{1}{2}$.
$\lambda = 3$, $\mu = 1\frac{2}{9}$, $\nu = 1\frac{1}{12}$; $\omega = 1\frac{1}{3}$;
$a = 12$, $b = 5$, $c = 21$, $d = 28$; $x = 16\frac{1}{3}$, $y = 27$, $z = 24$.

A. P. ROLLETT.

THE PRINCIPAL PARTS OF THE GENERAL CONIC.

By H. V. Styler.

An article by Lawrence Crawford in the *Gazette* for February 1934 (see also *Gazette*, February 1935) pointed out that the textbooks rarely make any attempt to relate coordinates of foci to the equations of corresponding directrices, etc. The article went on to show how this could be done.

In what follows an entirely different method is adopted. The method leads to actual formulae for all the principal parts of the conic.

I. The Central Conic.

The gradients of the axes, obtained from

$$(x^2-y^2)/(a^2-b^2)=xy/h,$$

are given by $$hm^2+(a-b)\,m-h=0.$$

Of the two solutions of this equation, one, referred to in what follows as μ, *has the same sign as* $\Delta\,.\,h$ where

$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

This is the gradient of the major axis. We are thus able, if we wish, to avoid altogether, in numerical work, the imaginary foci. Alternatively, the criterion enables us to associate either pair of foci with the directrices to which they correspond.

The following formulae will be established :

1. The (real) foci :

$$\frac{1}{C}\left\{G\pm\sqrt{\left(\frac{\Delta h}{\mu}\right)}\right\},\quad \frac{1}{C}\left\{F\pm\mu\sqrt{\left(\frac{\Delta h}{\mu}\right)}\right\},$$

the upper signs, or the lower, being taken together. These are best remembered in the form (*see* 3 below) :

$$Cx-G=\pm\sqrt{(\Delta h/\mu)},$$
$$Cy-F=\pm\mu\sqrt{(\Delta h/\mu)}.$$

2. The *corresponding* directrices are then

$$\xi+\mu\eta=\mp\sqrt{(\Delta\mu/h)},$$

where $$\xi\equiv ax+hy+g,\quad \eta\equiv hx+by+f.$$

3. Equation of major axis : $\mu(Cx-G)=(Cy-F)$.

4. Equation of minor axis : $(Cx-G)+\mu(Cy-F)=0$.

5. Eccentricity : $e^2=1-\dfrac{a+\mu h}{b-\mu h}$.

(Cf. *Gazette*, February 1934, p. 44, where $\rho_1=1/\mu$).

6. The major semi-axis : $\alpha^2=-\Delta/C(a+\mu h)$.

7. The minor semi-axis : $\beta^2=-\Delta/C(b-\mu h)$.

Proofs of these formulae.

6, 7. These follow readily from the equation of the conic referred to its centre as origin, namely

$$ax^2+2hxy+by^2+(\Delta/C)=0.$$

For, if $\mu=\tan\theta$, the point $(\alpha\cos\theta, \alpha\sin\theta)$ lies on the conic, and hence

$$\begin{aligned}\alpha^2&=\frac{-\Delta/C}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}\\&=\frac{-(\Delta/C)(1+\mu^2)}{a+2h\mu+b\mu^2}\\&=\frac{-(\Delta/C)(1+\mu^2)}{(a+h\mu)(1+\mu^2)}\\&=\frac{-\Delta}{C(a+h\mu)},\end{aligned}$$

for since $h\mu^2+(a-b)\mu-h=0,$

we have $b\mu^2+h\mu=\mu^2(a+h\mu).$

Similarly for β^2.

5. $e^2=1-\dfrac{\beta^2}{\alpha^2}=1-\dfrac{a+\mu h}{b-\mu h}.$

1. The coordinates of the foci, relative to the centre, can then be obtained. The following method, though longer, is, however, more interesting.

From the familiar identity

$$\begin{aligned}&(A+k)l^2+2Hlm+(B+k)m^2+2Gl+2Fm+C\\&\equiv C(lx_1+my_1+1)(lx_2+my_2+1),\end{aligned}$$

where (x_1, y_1), (x_2, y_2) are a pair of foci, and

$$\begin{vmatrix}A+k & H & G\\ H & B+k & F\\ G & F & C\end{vmatrix}=0,$$

we obtain

$$C(x_1+x_2)=2G,\quad Cx_1x_2=A+k.$$

Thus x_1 and x_2 are the roots of

$$Cx^2-2Gx+(A+k)=0,$$

that is,
$$\begin{aligned}(Cx-G)^2&=G^2-CA-Ck\\&=-\Delta b-Ck.\end{aligned}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now the determinant for k gives

$$Ck^2+\Delta(a+b)k+\Delta^2=0;$$

that is,
$$(Ck/\Delta)^2+(a+b)(Ck/\Delta)+C=0.\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Also the gradients of the axes are given by

$$hm^2+(a-b)\,m-h=0$$

or
$$(hm-b)^2+(a+b)(hm-b)+C=0. \quad \text{.....(iii)}$$

Comparing (ii) and (iii) we have

$$Ck/\Delta=hm-b, \text{ for } k_1, m_1, \text{ and } k_2, m_2;$$

that is,
$$Ck+\Delta b=\Delta hm.$$

Equation (i) now reads

$$(Cx-G)^2=-\Delta mh.$$

For the *real* foci we choose $m=-1/\mu$, and then

$$(Cx-G)^2=\Delta h/\mu,$$

whence follow the given formulae.

2. Directrices. The polar of (x_1, y_1) is

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0.$$

Substituting $x_1=(G\pm\lambda)/C, \quad y_1=(F\pm\mu\lambda)/C,$

where λ stands for $\sqrt{(\Delta h/\mu)}$, we have for the directrices

$$\frac{x}{C}(aG+hF+gC)+\frac{y}{C}(hG+bF+fC)+\frac{1}{C}(gG+fF+cC)$$
$$\pm\frac{\lambda}{C}\{x(a+h\mu)+y(h+b\mu)+g+f\mu\}=0,$$

that is,
$$\xi+\mu\eta=\mp\frac{\Delta}{C}\cdot\frac{C}{\lambda},$$

or
$$\xi+\mu\eta=\mp\sqrt{(\Delta\mu/h)}.$$

Numerical example for the ellipse.

$$5x^2+4xy+2y^2-2x+4y+4=0.$$

$$\Delta=\begin{vmatrix}5&2&-1\\2&2&2\\-1&2&4\end{vmatrix}; \quad \begin{matrix}G=6\\F=-12\\C=6\end{matrix} \quad ; \quad \begin{matrix}Gg=-6\\Ff=-24\\Cc=\ 24\\\hline\Delta=-6\\\Delta h=-12\end{matrix}$$

Thus μ is negative, and since it is a root of

$$2\mu^2+3\mu-2=0$$

we have
$$\mu=-2.$$

1. The foci:
$$6x-6=\pm\sqrt{6},$$
$$6y+12=(-2)(\pm\sqrt{6}).$$

Thus the foci are $\left(1\pm\frac{1}{\sqrt{6}},\ -2\mp\frac{2}{\sqrt{6}}\right)$.

Incidentally the centre is $(1, -2)$.

2. The corresponding directrices are $\xi - 2\eta = \mp\sqrt{6}$,
that is, $x - 2y = 5 \mp \sqrt{6}$.

3. The major axis is $-2(x-1) = y+2$,
or $2x + y = 0$.

4. The minor axis is $(x-1) - 2(y+2) = 0$,
or $x - 2y = 5$. (*Cf.* 2 above.)

5. Eccentricity: $e^2 = 1 - \dfrac{5-4}{2+4} = \dfrac{5}{6}$.

Thus $e = \sqrt{\frac{5}{6}}$.

6. The major semi-axis:

$$\alpha^2 = -\frac{-6}{6(5-4)} = 1 \quad \text{and} \quad \alpha = 1.$$

7. The minor semi-axis:

$$\beta^2 = -\frac{-6}{6(2+4)} = \tfrac{1}{6} \quad \text{and} \quad \beta = \sqrt{\tfrac{1}{6}}.$$

II. The Parabola.

The corresponding results for the parabola are as follows:

1. Focus: $2Gx - A = -\Delta/(a+b)$,
$2Fy - B = -\Delta/(a+b)$.

2. Vertex: $2Gx - A = -\Delta a/(a+b)^2$,
$2Fy - B = -\Delta b/(a+b)^2$.

3. Directrix: $2Gx + 2Fy = A + B$.

4. Tangent at vertex:

$$2Gx + 2Fy = A + B - \frac{\Delta}{a+b}.$$

5. Semi-latus rectum: $\sqrt{\{-\Delta/(a+b)^3\}}$.

6. Axis. This is best obtained by using the facts that it passes through the focus and that its gradient is known, but it can be written as

$$\frac{2Gx - A}{b} + \frac{\Delta}{b(a+b)} = \frac{2Fy - B}{a} + \frac{\Delta}{a(a+b)}.$$

Proofs of the above results.

1. For the focus, from the identity

$$Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + k(l^2 + m^2)$$
$$\equiv (lx_1 + my_1 + 1)(2Gl + 2Fm), \qquad (C = 0)$$

where (x_1, y_1) is the focus, we have

$$2Gx_1 = A + k,$$
$$2Fy_1 = B + k,$$

and
$$\begin{vmatrix} A+k & H & G \\ H & B+k & F \\ G & F & 0 \end{vmatrix} = 0,$$

so
$$k(-F^2-G^2)+\Delta^2=0,$$

and since $-F^2=\Delta a$, $-G^2=\Delta b$, we have
$$k=-\Delta/(a+b),$$

whence the formulae for the focus.

3. The directrix, the degenerate form of the director circle, requires no comment.

2, 4. The tangent at the vertex, being halfway between the directrix and the parallel to it through the focus, namely
$$2Gx+2Fy=A+B-\frac{2\Delta}{a+b},$$

has the equation given above, whence also the coordinates of the vertex.

5. The semi-latus rectum may be obtained in the same way, but is more easily obtained from the fact that $\Delta/(a+b)^3$ is known to be invariant, and in the case of $y^2=4px$ has the value $-4p^2$: hence
$$2p=\sqrt{\{-\Delta/(a+b)^3\}}.$$

6. The point at infinity on the axis is $(G:F:0)$. The axis is therefore parallel to
$$x/G=y/F,$$

that is, to
$$Gx/G^2=Fy/F^2,$$

or, since
$$-G^2=\Delta b, \quad -F^2=\Delta a,$$

to
$$2Gx/b=2Fy/a.$$

Since it passes through the focus, its equation is
$$\frac{2Gx-A}{b}+\frac{\Delta}{b(a+b)}=\frac{2Fy-B}{a}+\frac{\Delta}{a(a+b)}.$$

N.B. In the case of the parabola, since $ab=h^2$, a and b have necessarily the same sign and Δ has the opposite sign; the gradient of the axis has the same sign as Δh as in the case of the central conic.

Numerical example for the parabola.
$$4x^2-4xy+y^2-4x-8y+16=0.$$

$$\Delta=\begin{vmatrix} 4 & -2 & -2 \\ -2 & 1 & -4 \\ -2 & -4 & 16 \end{vmatrix}; \quad \begin{array}{ll} G=10; & Gg=-20 \\ F=20 & Ff=-80 \\ C=0 & Cc=0 \\ & \hline \Delta=-100 \\ & \Delta h=+200 \end{array}$$

Thus μ is positive, and being a solution of

$$-2m^2+3m+2=0$$

it has the value $+2$.

1. The focus : $20x-0=\frac{100}{5}$,
$40y-60=\frac{100}{5}$,

and the focus is (1, 2).

2. The vertex : $20x-0=\frac{4}{5}\,.\,20$,
$40y-60=\frac{1}{5}\,.\,20$,

and the vertex is $(\frac{4}{5}, \frac{8}{5})$.

3. The directrix : $20x+40y-60=0$,

or $$x+2y=3.$$

4. The tangent at the vertex :

$$20x+40y=60+20,$$

or $$x+2y=4.$$

5. The semi-latus rectum :

$$\sqrt{(100/5^3)}, \text{ that is, } 2/\sqrt{5}.$$

6. The axis : its gradient is 2, it passes through (1, 2), and hence its equation is $2x-y=0$. H. V. S.

1333. The philosopher's soul " withdraws itself as far as it can from all association and contact with the body and reaches out after truth by itself." With what results? Deprived of its nourishment, the soul grows thin and mangy, like the starved lion. Disgusted and pitying in the midst of our admiration, " Poor brutes! " we cry at the sight of such extraordinary and lamentable souls as those of Kant, of Newton, of Descartes. " Why aren't they given enough to eat? "—Aldous Huxley, *Do what you will*, p. 43 (London, 1929). [Per Mr. Frank Robbins.]

1334. In international politics, as in physics, the force of mutual attraction between two bodies is proportional to their respective masses. That is why Italy has got so little out of her allegiance to the Axis and why she looks likely to get nothing more substantial in the future except trouble.—*Daily Telegraph*, May 8, 1939. [Per Mr. H. N. Jones.]

1335. In older autogiro models, the rotor blades have been rotated while in a position of climb, that is, with the blades tilted from the horizontal up into the air. It has been found that if the blades are rotated while kept horizontal and parallel with the ground, greater revolutions result. Kinetic energy is stored. During this racing of the rotor blades, they are connected to a clutch. The energy stored is greater than the weight of the aircraft. When, therefore, the blades are released from the clutch, and are tilted to an angle of attack in relation to the air, the autogiro must leap upwards. The stored kinetic energy is released, and, being greater in volume than the weight of the aircraft, ascent is inevitable. The pilot cannot prevent it. —From an Air Display Programme. [Per Mr. M. Milford.]

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

SIR,—We owe a debt to Mr. Tuckey for his reply to Mr. Blackwell. Many of us, when we read criticisms of mathematical teaching, probably gain from them; but it is important to distinguish the true from the false, and surely Mr. Tuckey is right when he denies a number of the author's assumptions?

Criticisms such as those of Mr. Blackwell are still abundant. I do not deny their value. In a paper I read some time ago to a local branch of our association on the strictures contained in the Spens report, I pointed out that in no particular did that report say anything that had not been already urged by the teaching committee of the Mathematical Association. I suggested that the neglect of what had been done, by the author of the Spens report, was unfortunate and misleading. It gave me great satisfaction to read, a few months later, a declaration by our committee asserting the same thing.

And now Mr. Blackwell pursues a similar theme, and Mr. Tuckey answers him. I would support the latter by saying that in my judgment no subject has in the last thirty or forty years so largely improved its teaching methods. In my own schooldays the subject was still taught by many on old-fashioned lines; the quickening influence of Mr. Tuckey at that particular school was only just beginning to make itself felt. I remember well how a colleague of his, with a considerable reputation as a mathematical teacher, heard a class of forty boys say in early morning school a number of Euclidean propositions. It fell to me to quote the enunciation of Euclid I. 4, and to lose all my marks, with a serious admonition, for omitting the famous "each to each". I could tell similar stories, and so could others. Surely the gulf that separates that time from to-day is immense? And yet Mr. Blackwell suggests that we are still in many respects in the dark ages!

But Mr. Tuckey is perhaps at his best when he denies the insinuation that Mathematics is an unpopular subject. Often has it been mentioned to me as one of the more popular subjects of the curriculum. This was not true forty years ago. I certainly know schools where it is true to-day. There are no doubt many commonplace teachers in Mathematics, as in all subjects, but the general level has risen, and in consequence so has the popularity of the subject. Mr. Tuckey tells us "if the boys don't enjoy the work, it is your fault". I am confident this is true of Mathematics.

No one would deny the possibilities of further improvement. Professor Hogben and Mr. Westaway have both been quoted by Mr. Blackwell. These brilliant teachers can clearly do what most of us cannot, but I suggest they have provided something at which most of us are aiming, and while criticism is often helpful, pessimism is completely unjustified. Yours faithfully, W. F. BUSHELL.

Birkenhead School.

MATHEMATICAL NOTES.

1483. *On Note* 1471.

I have received so many solutions of this problem—over twenty, and no two quite alike—that instead of writing to correspondents individually, I should like to express my gratitude here. Perhaps I can best do so by explaining how the problem arose, and indicating those solutions which seem to me most useful for practical purposes.

In designing small boats I draw the half-deck plan (and certain other lines) as parabolic envelopes. That is to say, I am given two points, P and Q, on a parabolic arc; I know that the arc must touch at a given point O a given straight line which is parallel to the centre line of the boat; and I have to find R, the intersection of the tangents to the required arc at P and Q.

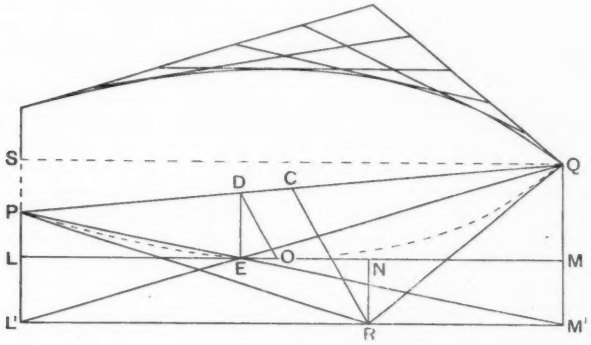


FIG. 1.

PL, QM, being the ordinates of P and Q, let them meet at L' and M' a straight line parallel to LM at a distance from it equal to the geometric mean of PL, QM. Then R lies on $L'M'$.

CONSTRUCTION I (due to Mr. P. V. MacKinnon). PM', QL' will intersect, say at E, on LM. Let C be the midpoint of PQ, and let DE, at right angles to LM, meet PQ at D. Then CR is parallel to DO.

II (due to Mr. R. Walker). Let PO, QO, meet $L'M'$ at P' and Q'. Then R is the midpoint of $P'Q'$. This is the simplest construction, but is less convenient than No. I because it prolongs the figure.

III (due to Mr. R. C. Lyness). A line concurrent with PQ and LM (so that it may bisect PL, QM) meets OP and OQ at P'' and Q''. PQ'', QP'' meet at X, and R lies on XO.

IV. Several correspondents point out that if RN is the ordinate of R, $ON = (OL \cdot QM - OM \cdot PL)/2RN$. Therefore to find ON we have to find the difference of two fourth proportionals which have a common term. This can be neatly done in a single figure, where

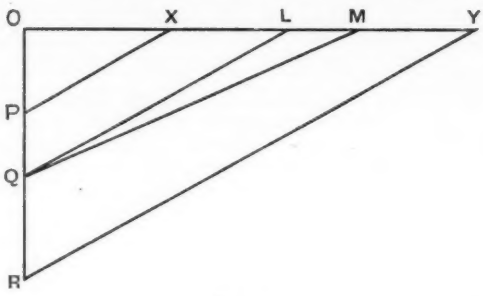


FIG. 2.

OP, OQ, and OR are the ordinates of P, Q and R, OL, OM are half the abscissae of P and Q, and XY is the abscissa of R.

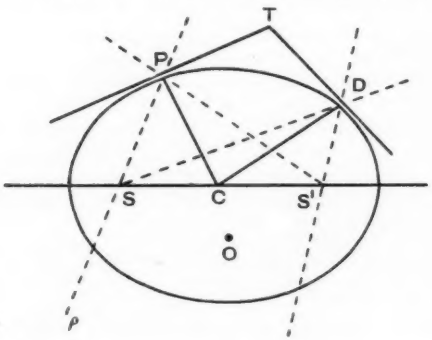
In the accompanying figure for Construction I, I have also shown a reflection of the required parabolic envelope about SQ, the centre line of the boat.

The cross-sections of such a boat are drawn as hyperbolic envelopes collated by further geometrical constructions; and it may interest your readers to know that one of my pupils, who is keen on boats but can't do the simplest arithmetic, learnt to draw hyperbolic envelopes merely by watching me draw the full-sized cross-sections of a boat we were building. I therefore taught him the whole method which I had not previously thought him capable of learning; and he has recently built himself a first-rate sailing dinghy thus designed by himself.

If any reader would care to see a simple example of such a design I should be pleased to lend him one. A. ROMNEY GREEN.

1484. *Unequal lines are sometimes equal.*

Let CP, CD be conjugate diameters of an ellipse, such that the focal distances SP, $S'D$ are admittedly unequal. Let it also be



admitted—see, *e.g.*, Smith, *Geometrical Conics* (1904), p. 114, No. 8—that the lines $SP, SD, S'P, S'D$ touch a circle. Let O be the centre of this circle, and let the tangents to the ellipse at P, D meet at T.

Then OP bisects $\angle SPS'$, the angle between the tangents to the circle, and TP bisects the exterior angle SPS', so $\angle OPT$ is a right angle. Again, OS bisects $\angle pSD$ between the tangents to the circle, and TS bisects $\angle PSD$, so $\angle OST$ is a right angle. Similarly, $\angle ODT$ and $\angle OS'T$ are right angles. Hence T, P, S, O, S', D are concyclic. But $\angle STP = \angle S'TD$ in relation to the ellipse and so, for the time being, $SP = S'D$. T. R. DAWSON.

1485. *Comments on Note* 1464.

(i) Figures 2 and 4 of Dr. Langford's note *do* extend to the case of nine figures fitting together to form a larger similar figure.

(ii) A three-dimensional analogy to his Fig. 1 might be provided by the figure obtained by constructing cubes on the faces $ABCD$, $ABFE$, $ADHE$ of the cube $(ABCD, EFGH)$. The solution is, however, very simple, and not analogous to the two-dimensional case.

(iii) The figures shown below are curious, as 144 of them can be fitted into a similar figure, but not, I believe, any smaller number.

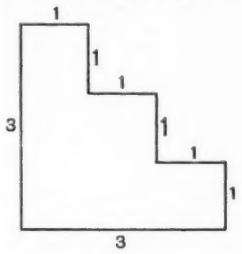


FIG. 1.

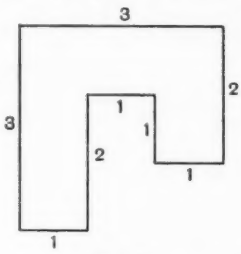


FIG. 2.

R. SIBSON.

1486. *Notes on differential equations.* 1. *Solution of linear differential equations by division.*

The solution of $(D^2+1)y=0$ can be written

$$\begin{aligned} y &= \frac{1}{D^2+1}\cdot 0 \\ &= \left(1-\frac{1}{D^2}+\frac{1}{D^4}-\ldots\right)\frac{1}{D^2}\cdot 0 \\ &= \left(1-\frac{1}{D^2}+\frac{1}{D^4}-\ldots\right)(A+Bx) \\ &= A\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)+B\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right) \\ &= A\cos x + B\sin x. \end{aligned}$$

The solution can be set out as a long division sum, and, as we shall see in further examples, this method has several advantages. The part of the solution which leads to $A\cos x$ is set out below.

$$
\begin{array}{rl|l}
 & & A-\frac{Ax^2}{2!}+\frac{Ax^4}{4!}-\dots \\
\hline
D^2+1 & 0 & \\
 & 0+A & \\
 & -A & \\
 & -A-\frac{Ax^2}{2!} & \\
\hline
 & \frac{Ax^2}{2!} & \\
 & \frac{Ax^2}{2!}+\frac{Ax^4}{4!} & \\
\hline
 & \dots\dots &
\end{array}
$$

At each step the next term is chosen so that when D^2 operates on this term it converts it into the dividend.

The same method can be applied with the operator $\Delta \equiv x\dfrac{d}{dx}$. Bessel's equation of zero order can be written

$$(\Delta^2+x^2)\,y=0,$$

and so

$$y=\frac{1}{\Delta^2+x^2}\cdot 0.$$

Setting this out as a long division sum,

$$
\begin{array}{rl|l}
 & & 1-\frac{x^2}{2^2}+\frac{x^4}{2^2\,.\,4^2}-\dots \\
\hline
\Delta^2+x^2 & 0 & \\
 & 0+x^2 & \\
 & -x^2 & \\
 & -x^2-\frac{x^4}{2^2} & \\
\hline
 & \frac{x^4}{2^2} & \\
 & \frac{x^4}{2^2}+\frac{x^6}{2^2\,.\,4^2} & \\
\hline
 & \dots\dots &
\end{array}
$$

For working this out the arbitrary constant is omitted, and this is of course only one solution. In this division we note that $\Delta x^r = rx^r$ so that when the dividend is $-x^2$ the quotient is $-x^2/2^2$. There is no other power of x except x^0 which makes $\Delta^2 x^r = 0$; consequently

there is no other solution in ascending powers of x. However, there is another solution of $\Delta^2 y = 0$, namely $\log x$. This gives $x^2 \log x$ as the first remainder, so that we have to find the function

$$\frac{1}{\Delta^2}(x^2 \log x).$$

This can be done by using the identity

$$\Delta^2\left(\frac{x^r \log x}{r^2} - \frac{2x^r}{r^3}\right) = x^r \log x,$$

whence

$$\frac{1}{\Delta^2}(x^r \log x) = \frac{x^r \log x}{r^2} - \frac{2x^r}{r^3}.$$

The complete division is set out below :

$$\log x - \left(\frac{x^2 \log x}{2^2} - \frac{2x^2}{2^3}\right) + \left(\frac{x^4 \log x}{2^2 \cdot 4^2} - \frac{2x^4}{2^2 \cdot 4^3} - \frac{2x^4}{2^3 \cdot 4^2}\right) - \cdots$$

$$\Delta^2 + x^2 \;\Big|\; \begin{array}{l} 0 \\ 0 + x^2 \log x \\ \quad -x^2 \log x \\ \quad -x^2 \log x - \dfrac{x^4 \log x}{2^2} + \dfrac{2x^4}{2^3} \\ \qquad\qquad \dfrac{x^4 \log x}{2^2} - \dfrac{2x^4}{2^3} \\ \qquad\qquad \dfrac{x^4 \log x}{2^2} - \dfrac{2x^4}{2^3} + \dfrac{x^6 \log x}{2^2 \cdot 4^2} - \dfrac{2x^6}{2^2 \cdot 4^3} - \dfrac{2x^6}{2^3 \cdot 4^2} \\ \qquad\qquad\qquad\qquad \cdots\cdots\cdots \end{array}$$

whence

$$y = \log x\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \cdots\right) + \frac{2x^2}{2^2} \cdot \frac{1}{2} - \frac{2x^4}{2^2 \cdot 4^2}\left(\frac{1}{2} + \frac{1}{4}\right) + \frac{2x^6}{2^2 \cdot 4^2 \cdot 6^2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right) - \cdots.$$

The division method is even more striking in cases where the term in $\log x$ is not the first term. When there are two powers of x differing by an integer which will each do as the first term of the quotient, there is no necessity to reject one of them as in the usual method of solution in series. We proceed with each in the usual way but with the larger one we find that at a certain stage in the division the quotient has to be of the form $x^r \log x$ instead of x^r. The solution of Bessel's equation of the first order is given below.

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1)y = 0.$$

Thus

$$\{(\Delta^2 - 1) + x^2\}y = 0$$

and

$$y = \frac{1}{\Delta^2 - 1 + x^2} \cdot 0.$$

(*a*) Solution without $\log x$.

$$\Delta^2 - 1 + x^2 \;\Big|\; \begin{array}{l} x - \dfrac{x^3}{2.4} + \dfrac{x^5}{2.4.4.6} - \cdots \\ \hline 0 \\ 0 + x^3 \\ \hline -x^3 \\ -x^3 - \dfrac{x^5}{2.4} \\ \hline \dfrac{x^5}{2.4} \\ \dfrac{x^5}{2.4} + \dfrac{x^7}{2.4.4.6} \\ \hline \cdots\cdots \end{array}$$

This is the solution $y = 2J_1(x)$.

(*b*) Solution containing $\log x$.

In finding this solution we shall need the value of

$$\frac{1}{(\Delta - a)(\Delta - b)} \cdot x^r \log x.$$

Now $(\Delta - a)(\Delta - b)\{x^r \log x\}$

$$= (\Delta - a)\{(r - b)x^r \log x + x^r\}$$
$$= (r - a)(r - b)x^r \log x + (2r - a - b)x^r.$$

Thus

$$\frac{1}{(\Delta - a)(\Delta - b)} x^r \log x = \frac{x^r \log x}{(r - a)(r - b)} - \frac{1}{(\Delta - a)(\Delta - b)} (2r - a - b)x^r$$
$$= \frac{x^r \log x}{(r - a)(r - b)} - \frac{(2r - a - b)x^r}{(r - a)^2 (r - b)^2}.$$

But when $r = a$ or $r = b$,

$$\frac{1}{(\Delta - a)(\Delta - b)} x^r = -\frac{x^r \log x}{2r - a - b}.$$

Both of these results are used in the division below.

$$\Delta^2 - 1 + x^2 \;\Big|\; \begin{array}{l} x^{-1} - \frac{1}{2}\log x + \frac{1}{2}\left\{\dfrac{x^3 \log x}{2.4} - \dfrac{6x^3}{2^2.4^2}\right\} - \cdots \\ \hline 0 \\ 0 + x \\ \hline -x \\ -x - \frac{1}{2}x^3 \log x \\ \hline \frac{1}{2}x^3 \log x \\ \frac{1}{2}x^3 \log x + \frac{1}{2}\left\{\dfrac{x^5 \log x}{2.4} - \dfrac{6x^5}{2^2.4^2}\right\} \\ \hline \cdots\cdots \end{array}$$

Proceeding in this way we get

$$y = x^{-1} - \tfrac{1}{2}x\log x + \tfrac{1}{2}\left\{\frac{x^3\log x}{2.4} - \frac{6x^3}{2^2.4^2}\right\}$$
$$+ \tfrac{1}{2}\left\{\frac{x^5\log x}{2.4^2.6} - \frac{8x^5}{2.4^3.6^2} - \frac{6x^5}{2^2.4^3.6}\right\}$$
$$+ \tfrac{1}{2}\left\{\frac{x^7\log x}{2.4^2.6^2.8} - \frac{10x^7}{2.4^2.6^3.8^2} - \frac{8x^7}{2.4^3.6^3.8} - \frac{6x^7}{2^2.4^3.6^2.8}\right\} + \dots$$
$$= x^{-1} - \left\{\frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \dots\right\}\log x$$
$$- \frac{6}{2.4}\cdot\frac{x^3}{2^2.4} + \left(\frac{6}{2.4} + \frac{8}{4.6}\right)\frac{x^5}{2^2.4^2.6}$$
$$- \left(\frac{6}{2.4} + \frac{8}{4.6} + \frac{10}{6.8}\right)\frac{x^7}{2^2.4^2.6^2.8} + \dots .$$

The noteworthy point about this solution is that there is no term in x, the place of this term being taken, as it were, by $-\frac{1}{2}x\log x$. On rearrangement it is found that this solution is equal to

$$-Y_1(x) - \tfrac{1}{2}J_1(x),$$

where $Y_1(x)$ is Neumann's Bessel function of the first order.

H. V. L.

1487. *Notes on differential equations.* 2. *Methods of discussing the behaviour of the solution by studying the differential equation.*

From the equation

$$\frac{d^2y}{dx^2} + y = 0,$$

we see that if a solution starts from, say, $y = +b$ at $x = 0$, d^2y/dx^2 is negative at $x = 0$ and so dy/dx is decreasing at $x = 0$ and, since $d^2y/dx^2 = -y$, dy/dx continues decreasing until $y = 0$. Then d^2y/dx^2 changes sign and dy/dx increases until $y = 0$ again; after this the process is repeated. This shows that the graph of y is a wave with points of inflexion on Ox, and each wave must be the same shape because d^2y/dx^2 has the same value at points at the same height on different waves. The solution is therefore an oscillation with a constant period.

Bessel's equation of zero order can be written

$$(\Delta^2 + e^{2\theta})y = 0,$$

where $$\Delta = x\frac{d}{dx} = \frac{d}{d\theta}, \quad \text{and} \quad x = e^{\theta}.$$

By the same argument as above we see that the solution of this equation is also a wave, but in this case for two points on different waves having the same ordinate y but abscissae θ_1 and θ_2 ($\theta_2 > \theta_1$),

$$\left|\frac{d^2y}{d\theta_2^2}\right| = |e^{2\theta_2}y| > |e^{2\theta_1}y| = \left|\frac{d^2y}{d\theta_1^2}\right|.$$

Thus $dy/d\theta$ is decreasing at a quicker rate at $\theta=\theta_2$ than at $\theta=\theta_1$. Hence, considered as a function of θ, y is a periodic function with oscillations of a decreasing period and decreasing amplitude. When the variable is changed back to x, which means in effect a change to an exponential scale on the θ axis, it still follows that the waves have decreasing amplitude but we cannot draw any conclusion about the period because the change to the new scale stretches out the curve horizontally in a greater and greater ratio as θ increases.

We give one further example of this method by applying it to a special case of Riccati's equation. The general case can be discussed similarly. Suppose

$$\frac{dy}{dx}+4y^2=9x^3, \quad \text{and} \quad y=1 \text{ at } x=0.$$

Then
$$\frac{dy}{dx}=9x^3-4y^2,$$

and so dy/dx is negative at $x=0$ and remains negative until $y=\frac{3}{2}x^{3/2}$. Thus y decreases from 1 at $x=0$ and so its graph is bound to cut the graph of $y=\frac{3}{2}x^{3/2}$ and at this point it will be horizontal.

For greater values of x, dy/dx will be positive and y will increase. It is impossible for the graph of y to cross the graph of $\frac{3}{2}x^{3/2}$ again owing to the fact that dy/dx vanishes at such a point. But if we put $y=\frac{3}{2}x^{3/2}+z$ and assume z to be small we get approximately for large x,

$$\tfrac{3}{2}x^{\frac{1}{2}} = -3x^{\frac{3}{2}}z$$

and
$$z=-1/2x.$$

Thus our assumption that z is small is justified and so the graph of y approaches the graph of $y=\frac{3}{2}x^{3/2}$ at $x=+\infty$ after having crossed it at one finite value of x. H. V. L.

1488. *Notes on conics.* 4: *The conic as a locus of ideal points.*

The elementary view of the relation of the eccentric circle to the conic is that if p on the eccentric circle of O corresponds to P on the conic, then SP is parallel to pO and OP and pS cut on the directrix s. Let us be content with the second half of the theorem, and express it rather differently: If p on the circle corresponds to P on the conic, and if Sp cuts s in R, then RO passes through P. This is a construction by which, given p, we find the line joining O to P without first finding P. If we apply the construction to a number of different centres in the plane, using on each eccentric circle the point that corresponds to P, we have a star of concurrent lines whose centre is P. In order to add lines to this star, we have only to be able to recognise corresponding points on different eccentric circles by some criterion which does not introduce the point on the conic. This is easy enough. Since the radii of eccentric circles are

arithmetically proportional to the distances of their centres from the directrix, the point in which a line O_1O_2 cuts s is one of the centres of similarity of the eccentric circles of O_1, O_2; there is of course a second centre of similarity, but as far as I know this second centre is of no significance in the theory of the conic, and we may call the centre on s the relevant centre. Points on the two circles which correspond to the same point on the conic correspond also in the similarity which has the relevant centre.

We can now reverse our whole outlook, for the star is developed from a point p on the eccentric circle of one point O, without reference to the conic at all. We remark, however, that whereas if we begin from P the point p cannot turn up on the directrix, if we are to begin from p we shall of course not impose any restriction but shall take p on the directrix if we can and see what happens. What happens is that if p is a point which the eccentric circle of O has in common with s, then RO coincides with pO, the corresponding point on any other eccentric circle is also on s, and the star of concurrent lines is replaced by a family of parallel lines. Thus it is inevitable that we should regard a family of parallel lines as a configuration that sometimes replaces a family of concurrent lines, and for economy of language we must either introduce a new term to include families of both kinds, or extend the meaning of concurrence to include parallelism. Both courses have been adopted. Pascal, in his *Essay pour les Coniques*, speaks* of an " ordonnance de lignes "; some writers, restricting " pencil " to a family of concurrent lines, have used " sheaf " in an inclusive sense; more familiar in elementary books is the talk of parallel lines as lines " concurrent at infinity ". Each course has its advantages, and each its disadvantages: if we use a new word we sacrifice all the associations of the word which it replaces; if we extend the connotation of an old word we risk forgetting that any change has really been made. With " concourse " or " sheaf " we feel that we have a new subject to learn, but if lines are " concurrent at infinity " we begin trying to imagine and to locate a point of intersection, and we are soon involved in paradoxes, for nothing is clearer than that the parabola does not " really " bend down and return to its axis in some region indefinitely remote.

There is another serious difficulty of language. We may appreciate that theorems about families of lines are subject to fewer exceptions than theorems about points, but the point is our primary concern, and we do not want to change as it were the dimensions of our vocabulary in such a way that elementary theorems are no longer recognisable. This consideration is the key to the solution of the problem; *the family of lines must be called a point*, and since some qualification is required, the name actually used is " ideal point ". An ideal point is a family of lines, all concurrent or all

* D. E. Smith (*Source Book*, p. 326, f.n.) says that the definition is taken almost word for word from Desargues, but I have not the original available for discovering a precise reference.

parallel ; if the lines are concurrent, the ideal point is said to be accessible, if the lines are parallel, the ideal point is said to be inaccessible or at infinity, but in either case the constituents of the ideal point are just ordinary lines. With each ordinary point of the plane can be associated the accessible ideal point of which it is the centre. Any theorem about ordinary points can be reworded as a theorem about accessible ideal points, but as a rule this is a gratuitous complication. "As a rule", but if the theorem presents exceptional cases, it is then worth while to translate the theorem, with a view to discovering whether the corresponding theorem about ideal points is true without exception, in which event the cases which involve inaccessible points can be retranslated as theorems involving parallel lines.

We can now return to the correlation of the conic with an eccentric circle. There is always a point of the circle to correspond to a point of the conic, but the converse is not true ; if the conic is an ellipse the correlation is complete, but if the conic is a parabola or a hyperbola the correlation breaks down at one point or at two points of the circle. We can, however, relate the circle to a family of ideal points instead of to a family of ordinary points. The accessible ideal points are those whose points of concurrence compose the conic, in the elementary sense, and the ideal points are inaccessible just when the elementary correlation fails, and just because the elementary correlation fails. Far from being an artificial convention, it is the most natural thing in the world to replace the ordinary conic by an ideal conic which is an aggregate, or we may as well say a locus, of ideal points. Not only is the correspondence of the ideal conic with an eccentric circle complete, but the Boscovich construction is actually a direct construction, in the only sense in which construction of an ideal point is possible, for the ideal points which compose the ideal conic. E. H. N.

1489. *Notes on conics. 5 : theorems as constructions.*

I wonder if a taint of inferiority clings to the idea of a construction, in supposed contrast with a theorem, from the early stages when intuitive and practical geometry precede systematic and deductive work. The contrast is usually artificial. The theorem of equal intercepts between a hyperbola and its asymptotes, for example, is a construction for as many points as we wish on the hyperbola of which the asymptotes and one point are given, and it may be invoked for this purpose in analytical geometry when a hyperbola is to be drawn from its equation.

Pascal's theorem we are ready enough to present as a construction. It is at the beginning of our study of conics that we are most timid. Almost the first theorem we learn is that if a chord PQ cuts the directrix corresponding to the focus S in R, then SR is a bisector of the angles between SP and SQ. How many of us express this theorem alternatively, or drive home its implication emphatically : *Given focus S and directrix s and one point P on the conic, to find the*

second point in which a line l through P cuts the conic, reflect SP in the line joining S to the point of intersection of l and s ; the reflection cuts l in the point required. Since S and P are fixed, obvious constructions for the reflected line are by means of a circle on SP as diameter and by means of a circle through P with S for centre. But there is a third method. If l cuts s in R, and if the circle round P through S cuts the line SR again in the point q, then qP and SP are equally inclined to SR, and therefore SQ is parallel to qP. This is the eccentric circle construction, in the simple form which it assumes when the centre of the circle is a point on the conic.

Is there not an air of patronage in our attitude to the eccentric circle, for the same reason? The eccentric circle of O is introduced as a device for finding the points in which a line through O cuts the conic or for constructing the tangents from O, and because we think of it as "only" a practical device we do not acquire the habit of using it for theoretical purposes. Except for demonstration of the rectangle theorem we ignore it completely. Boscovich himself made none of our mistakes. "Plurima quidem ex ea [constructio] inferri possunt theoremata, & pleraque utilissima ac iterum foecunda." (*Elementorum Universæ Matheseos*, III (1754), p. 52). The centre of his circle is at first an arbitrary point, unrelated to the lines whose intersections with the conic he wishes to consider, but in due course he makes full use of the power of choosing a centre appropriate to the problem in hand.

For an example of the services the eccentric circle is not encouraged to render, note first that if p, q on any eccentric circle correspond to P, Q on the conic, then although it is only if O is on PQ that pq is identical with SR, in any case pq is parallel to SR. If now O is the intersection of the tangents at P, Q to the conic, Sp, Sq are the tangents from S to the circle, and SP, SQ, being parallel to pO, qO, are equally inclined to SO as well as to SR. But there is now no temptation to evade the question whether SO might not sometimes coincide with SR, a question which to the scrupulous teacher is tiresome : SO is perpendicular to pq, and that is that.

E. H. N.

1490. *On Note* 1471.

Note 1471 is concerned with a parabola passing through two fixed points P, Q and touching a given straight line LM in a given point O. A geometrical construction is required for PR, QR, the tangents at P, Q. PL, QM, RN are drawn perpendicular to LM.

The result can easily be worked out by elementary coordinate geometry. Taking O as origin and OM as axis of x, the equation to the parabola is of the form $(x-\lambda y)^2=cy$; where, if $PL=h$, $QM=k$, $LO=a$, $OM=b$, the constants are determined by the condition that the parabola passes through the points $(-a, h)$, (b, k). The equations to the tangents at these points will be satisfied by the coordinates of R, $(X, -Y)$ say. We thus have four equations involving λ, c, X, Y from which, after a little teasing elimination,

we obtain the simple results

$$Y^2 = hk,\ XY = \tfrac{1}{2}(ak - bh);$$

i.e. $$RN^2 = PL.QM;\ \ 2ON.NR = LO.QM - OM.PL,$$

on which a geometrical construction for the tangents is easily based.

The work, however, seems a bit tricky for such a simple geometrical result; and we really keep closer to the geometry if we use areal-tangential coordinates, referring the parabola to the triangle of tangents RBC, where B, C are the intersections of the tangents RP, RQ with LM.

The tangential equation to the parabola will be of the form $lqr + mrp + npq = 0$ where $l + m + n = 0$, since the line at infinity $(1, 1, 1)$ is a tangent. The points of contact with the sides of RBC will be $mr + nq = 0$, etc., which interpreted geometrically give

$$BO : OC :: m : n; \quad \ldots\ldots(1)$$

similarly
$$CQ : QR :: n : l,$$
$$RP : PB :: l : m.$$

Since $-l = m + n$, we have

$$PB : BR :: m : n; \quad \ldots\ldots(2)$$
$$RC : CQ :: m : n. \quad \ldots\ldots(3)$$

By (2) and (3)
$$PL : NR :: m : n;$$
$$NR : QM :: m : n.$$

Hence $RN^2 = PL.QM$, one of the results required.

Also
$$PL : QM :: m^2 : n^2.$$

Then again, from the figure,

$$BN = \{n/(n+m)\}\ .\ LN, \text{ by (2)};$$
$$NC = \{m/(n+m)\}\ .\ NM, \text{ by (3)}.$$

But
$$BO = \{m/(n+m)\}\ .\ BC,$$

therefore
$$= \{mn/(m+n)^2\}\ .\ LN + \{m^2/(m+n)^2\}\ .\ NM.$$

Subtracting from BN, above, we have

$$ON = \{n^2/(m+n)^2\}\ .\ LN - \{m^2/(m+n)^2\}\ .\ NM$$
$$= \{n^2/(m+n)^2\}\ .\ (a + ON) - \{m^2/(m+n)^2\}\ .\ (b - ON),$$

i.e.
$$2mn\ .\ ON = n^2a - m^2b;$$

whence $2ON.NR = LO.QM - OM.PL$, as before.

This is an interesting illustration of the way in which areals may be neater than cartesians.

PERCY J. HEAWOOD.

1491. *Visible Edges of an Icosahedron.*

When next your pupils make a regular icosahedron for you, try asking them how many of its edges can be seen at the same time.

(Assume that it is opaque and looked at with one eye only.) I have found that less than a sixth of the boys I teach give 20 for the answer ; but this is the maximum if the most favourable viewpoint is chosen.

From some directions 19 or 18 can be seen, and less if the observer is too near to the icosahedron or in the plane of one of its faces, so that some edges hide others.

This suggests the problem : If a regular icosahedron, small in comparison with the observer's distance, is looked at from a single point in a random direction, what are the probabilities that 20, 19 or 18 edges will be visible?

The full working is tedious, but an outline is given here : to amplify it would make a good exercise for a beginner in Spherical Trigonometry.

Fig. 1 represents the icosahedron as seen by one looking down on to T, its top point. The five lowest edges are omitted : the sides

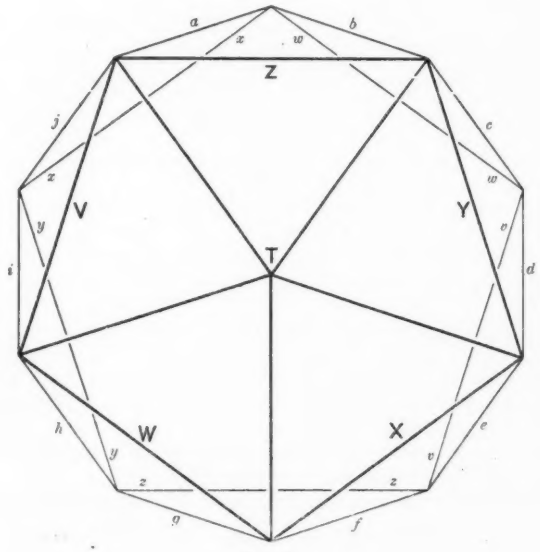


Fig. 1.

of the lower horizontal pentagon, z, y, x, w and v, at present out of sight, are shown by broken lines. The ten highest edges are shown by thick lines, and those of them which are opposite to z, etc., are lettered with capitals corresponding to their opposites.

The second figure represents part of a sphere, through the centre of which planes are drawn parallel to the faces of the icosahedron.

The outer spherical pentagon, whose angles are 120° each, is bounded by the intersections of the sphere with the five planes through the centre bisecting at right angles the edges meeting at T : its spherical excess is 60°, and its area is 1/12 of the sphere. If the icosahedron is placed at the centre of the sphere, then at all points inside this pentagon T is the nearest vertex of the icosahedron : also if the dimensions of the icosahedron are negligible in comparison with the radius, from all points inside the pentagon the ten edges drawn thick in Fig. 1 are visible.

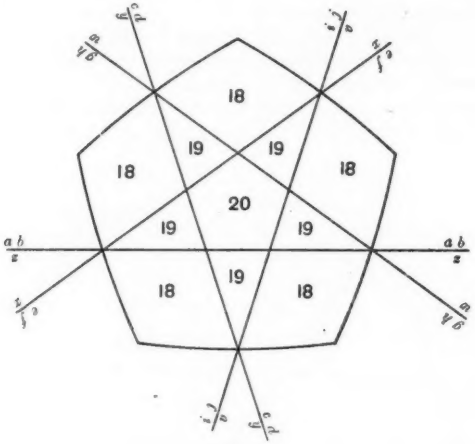


FIG. 2.

The sides of the inner pentagon are the intersections of the sphere with the planes parallel to the planes abZ, etc. The letters written on each side of these great circles mean that from points on one side the edges a and b are visible, but not z, from points on the other side z is visible, but not a or b.

From this it is easy to write in each area the number of edges which can be seen from points in that area.

The perpendicular on to a side of the inner pentagon from its centre

$$= \text{the inclination of the plane } abZ \text{ to the vertical}$$
$$= \cot^{-1}(3+\sqrt{5}).$$

Hence, by solving a right-angled triangle, each angle of the inner pentagon is $\cos^{-1}(-\frac{1}{3}) = 109° 28' 16{\cdot}4''$.

(This is the angle between two faces of the icosahedron which are not quite adjacent but meet corner to corner. It is of some interest to notice that it is also the angle between adjacent faces of a regular octohedron, between edges of a rhombic dodecahedron or bee's cell, or between the diagonals of a cube.)

Hence the spherical excess of the inner pentagon = 7° 21′ 22″ = (approx.) ·1226 or 13/106 of 60°.

Therefore the probability of seeing 20 edges = 13/106.

The isosceles triangles marked 19 have their vertices at the mid-points of the sides of the outer pentagon (which are also the intersections of the sphere with the radii through the mid-points of the edges meeting at T). Their vertical angles are $\sin^{-1}\frac{2}{3}$, the supplement of the angle between two adjacent faces of the icosahedron, that is, 41° 48′ 37·1″. This makes the spherical excess of each isosceles triangle to be 2° 52′ 04·3″, whence the probability of seeing 19 edges but not 20 is approximately ·2390 or 38/159.

Therefore the probabilities of seeing 20, 19 and 18 edges are

$$\frac{39,\ 76 \text{ and } 203}{318}.$$

The corresponding problem for the regular dodecahedron is not without interest. W. HOPE-JONES.

1492. *Triple conjunction.*

In this year of the triple conjunction of Jupiter and Saturn (which is said not to have occurred since A.D. 1683), it may be of a little interest to consider the simplified problem of three planets describing concentric circles in one plane about the sun as centre, in connection with such an event. Let the distances from the sun, in ascending order, be a, b, c; and suppose that, as seen from the innermost planet, the other two are simultaneously in opposition (as occurs this year for Jupiter and Saturn on November 3). What must be the condition satisfied that, after such a conjunction, the second planet may retrograde relatively to the third, as seen from the inner one? Then there will be another conjunction of the two outer as seen from the inner, in the same "year" of the inner planet; and similarly there will have been another earlier in the year, making three in all.

We suppose the law of nature to hold, by which the squares of the periods vary as the cubes of the distances, so that the angular velocities about the centre are proportional to $a^{-\frac{3}{2}}$, $b^{-\frac{3}{2}}$, $c^{-\frac{3}{2}}$. *Relatively* to the third we have angular velocities proportional to $a^{-\frac{3}{2}} - c^{-\frac{3}{2}}$, $b^{-\frac{3}{2}} - c^{-\frac{3}{2}}$; and the condition for the retrogression of the second with respect to the third, as seen from the first, will be

$$a(a^{-\frac{3}{2}} - c^{-\frac{3}{2}}) \div (c-a) > b(b^{-\frac{3}{2}} - c^{-\frac{3}{2}}) \div (c-b),$$

i.e. $$a(c-b)(a^{-\frac{3}{2}} - c^{-\frac{3}{2}}) > b(c-a)(b^{-\frac{3}{2}} - c^{-\frac{3}{2}}),$$

i.e. $$c(a^{-\frac{1}{2}} - b^{-\frac{1}{2}}) + ab(a^{-1} - b^{-1})c^{-\frac{1}{2}} - ab(a^{-\frac{3}{2}} - b^{-\frac{3}{2}}) > 0.$$

i.e. $$c^{\frac{3}{2}}(b^{\frac{1}{2}} - a^{\frac{1}{2}}) + (b-a)a^{\frac{1}{2}}b^{\frac{1}{2}} - c^{\frac{1}{2}}(b^{\frac{3}{2}} - a^{\frac{3}{2}}) > 0.$$

Factorising, we have

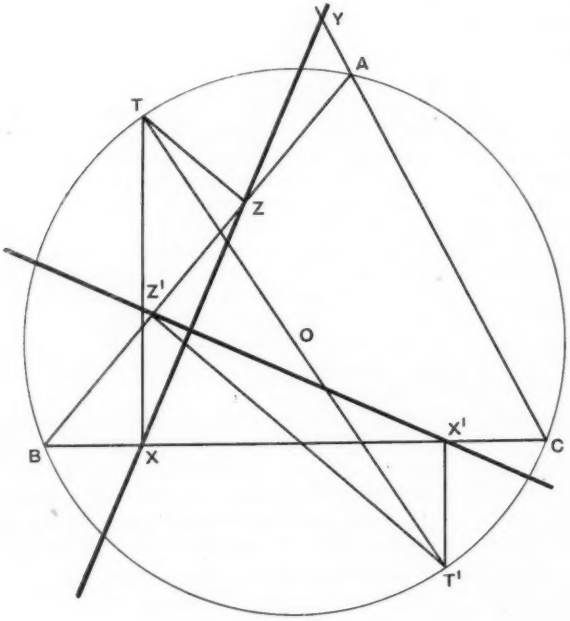
$$(c^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{\frac{1}{2}})(c^{\frac{1}{2}} - a^{\frac{1}{2}})(c^{\frac{1}{2}} - b^{\frac{1}{2}})(b^{\frac{1}{2}} - a^{\frac{1}{2}}) > 0,$$

which is necessarily satisfied in virtue of $c > b > a$.

Now $a = 92{\cdot}9$, $b = 483{\cdot}3$, $c = 886{\cdot}1$ are the values, in millions of miles, assigned as the mean distances from the sun of the earth, Jupiter and Saturn; but, as we have seen, when two outer planets are in opposition simultaneously, triple conjunction is a necessary consequence of the law that the squares of the periods are proportional to the cubes of the distances, and has no special connection with the actual dimensions of the orbits, at any rate when they approximate to the circular form. PERCY J. HEAWOOD.

1493. *An unexpected converse.*

In the case of a pair of Simson lines at right angles, as in the figure, since the perpendicular from O on AB bisects both AB and ZZ', and similarly for the sides BC, CA, it follows that the *Simson lines are isotomic conjugates to* ABC.



A partial converse of this, namely, that the isotomic conjugate of any Simson line is the Simson line at right angles to the first, is equally obvious by similar reasoning.

The complete converse, that *any pair of isotomic conjugate lines at right angles is a pair of Simson lines*, seems nevertheless somewhat unexpected. The direct proof is simple.

Given XZ, $X'Z'$ isotomic conjugate lines at right angles, let the perpendiculars at X, X' to BC and at Z, Z' to AB meet at T, T' as in the figure. It is obvious that TT' is bisected by the circumcentre O. Also

$$\begin{aligned}\angle TBT' &= \angle TBX + \angle X'BT' \\ &= \angle TZY + \angle X'Z'T' \\ &= \angle ZZ'X' + \angle X'Z'T' \\ &= \text{a right angle,}\end{aligned}$$

so TT' is a circumdiameter of ABC. T. R. DAWSON.

1494. *Products of vectors.*

Mr. Tuckey (Note 1462, *Gazette*, XXIV, p. 206) hits the nail on the head with characteristic efficiency, and it is in the spirit of his note that I would recall attention to a FOURTH product. In a plane in which a direction of angular measurement has been chosen, that is, in the plane of ordinary metrical plane geometry, the vectors (r, θ) and (r', θ') have as one of their products the scalar $rr' \sin(\theta' - \theta)$, which may be called their areal product.

In space of more than two dimensions, the areal product cannot be a scalar since it must be associated with a cyclical direction; fundamentally the product and the cyclically directed veeplane in association form a new concept, the bivector, which is recognisable in Euclidean space of any number of dimensions. In three dimensions only, direction of cyclical measurement in a plane can be associated with direction of linear measurement normal to the plane by a universal convention, and by associating the areal product immediately with the direction of the normal we evade the bivector and compel an ordinary vector to do its work. This is a trick peculiar to three dimensions, but we live in a three-dimensional world and are to be pitied rather than blamed if we are so dazzled by the technical success of our trick as to be blind to the very existence of directed magnitudes associated with planes.

Needless to say, no blame attaches to Hamilton in this matter: not only, as Mr. Tuckey hinted, is the idea of the vector product *as a product* incompatible with Hamilton's system, not only is Hamilton meticulous to the verge of tedium in insisting on logical distinctions between concepts with the same laws of operation, but the quaternion belongs essentially to space of three dimensions, and in the quaternion calculus any advantage to be derived from that limitation is a proper development. The quaternion is a mathematical dinosaur, an immense and powerful creature adapted so perfectly to one environment that it cannot exist or be modified to exist in any other. E. H. N.

1336. The gaining of insight into a phenomenon is just as indefinable a notion as a mathematical concept. But, as with a mathematical concept, we feel the better for it.—E. A. Milne, *Relativity, Gravitation and World-Structure*, p. 3.

REVIEWS.

A History of Geometrical Methods. By J. L. COOLIDGE. Pp. xviii, 451. 30s. 1940. (Oxford)

It seems at first sight a striking fact, as Professor Coolidge points out in the preface to this volume, that in spite of the appearance of numerous works on the history of mathematics in general there seems to have been no serious attempt to describe the development of geometrical methods since Chasles published his *Aperçu Historique* in 1837. An examination of the contents of the present volume make this a little less surprising. A general history of mathematics must be content to describe the scientific contributions of individual mathematicians in fairly general terms, a history of geometrical methods must penetrate rather more deeply and give a clear outline of at any rate the more important theories in the various branches of the subject. Few mathematicians, however, have the necessary comprehensive grasp of the many fields of modern geometry to attempt an adequate description of them all, and a history of geometrical methods must in addition deal in some detail with what is now usually called elementary geometry, a subject with a vast literature but with few exponents to-day. The compilation of a work of this sort is thus no mean task for any mathematician to undertake.

Professor Coolidge is probably better qualified than most people to tackle the bold undertaking of which this book is the outcome. His published works cover a wide range of geometrical topics, and prove their author to be a man with an eye for the historical aspects of the subject. Even so, the author has deliberately restricted the field he intends to cover. Thus no account of topology appears at all, and the work is, in effect, a study of the developments of three branches of geometry, synthetic, algebraic and differential, from the earliest times to the present day.

The reasons for the difficulty of writing a book of this nature make it almost equally difficult to review it. The three branches of the subject are to-day separated to such an extent as to make it practically impossible for a single mathematician to give an adequate critical account of the contents of this many-sided volume. I shall not attempt a task which is beyond my powers in the finite time in which decency compels me to complete this review, and shall accordingly confine my detailed criticism to one portion of the work.

The author interprets the phrase "earliest times" liberally by commencing with some entertaining remarks about the construction of spiders' webs. *Homo sapiens* appears on the scene on page 5, where an account is given of the geometry of the Babylonians. I confess to being struck by the fact that the formula for the volume of a truncated pyramid was used nearly four thousand years ago. Other ancient geometries are also described in the first chapter. The Greeks, of course, have a chapter to themselves, for here for the first time we come across the notion of mathematical proof. The author then traces the subsequent (and astonishingly slow) development of synthetic geometry in Western Europe, reaching its climax in the discovery of the non-Euclidean geometries. The story of the rise of projective geometry is next described, and the first section of the work closes with an account of the evolution of descriptive geometry.

Algebraic geometry next claims our attention. The use of coordinates for specifying the position of points in a plane or in space is traced from its beginning with Descartes to the introduction of coordinates of a generalised type in the nineteenth century; at the same time the evolution of more powerful methods of handling equations such as the theory of invariants is

also touched on. An interesting chapter follows dealing with enumerative geometry, with a description of the highly original (and highly contentious) methods introduced by Schubert. The author then turns to birational geometry, and gives a brief account of the theory of algebraic curves as developed by the Italian school of geometers. He next passes to the theory of algebraic surfaces. At this point I find his account unsatisfactory. The really interesting theorems in this field are much deeper than any which appear in the classical theory of curves; it would be out of place, in a work of this kind, to attempt to describe in any detail the underlying ideas involved in their proof. Professor Coolidge seems to me to try to invest the subject with a simplicity which in fact it does not possess. Thus, for instance, he describes very briefly the arguments used by Severi in his attempt to resolve the singularities of a surface, without pointing out that an essential step in Severi's argument has been discredited by Zariski. His approach to the postulation formula is not merely heuristic (which in a historical account it might well be), but bears not the slightest relation to the method by which the formula is in fact obtained. The ingenuous manner in which the irregularity of a surface appears as a non-negative integer on p. 224 makes one wish that life really was as simple as that. Unfortunately for the author, it cannot be shown that the number of conditions imposed on a surface by a curve which it is constrained to contain never exceeds the number given by the postulation formula, and the fact that this is in fact true for the canonical adjoints of a surface with a double curve requires special justification. I do not complain that this justification is not given, but I do object to the concealment of the fact that it is necessary. After all, others besides algebraic geometers will, I hope, read this volume. The author is not always precise in the enunciation of his theorems either: on p. 220 in the statement of Bertini's first theorem the possibility of multiple base-points has been overlooked.

Making every allowance for the difficulty of describing this intricate but fascinating theory in a dozen pages, I feel that the author has missed the mark. On p. 224 it is more than half suggested that an irregular surface is a sort of pathological phenomenon, whereas in fact it may be regarded in many respects as the natural generalisation of an irrational curve; and there is no hint given that the irregularity of a surface is in fact by far the most important of all its birational invariants. The theory of algebraic varieties of higher dimension is dismissed in a short paragraph. A critical account of this theory has never yet been seriously undertaken, and is badly needed. There is already a considerable literature of the subject, mostly of fairly recent date, but the foundations of the theory seem to me to rest on a number of assumptions, some only of which are explicit in the fundamental paper of Severi published in 1909. The result is a collection of interesting results, many of which cannot be held to be rigorously acquired, although I suppose that every worker in the field is convinced that they are true.

The next chapter of the book turns to the subject of projective geometry in higher space, to quaternions and the calculus of Grassmann, and to Plücker coordinates. Next follows a chapter tracing the development of the concept of a geometrical transformation from Möbius through Cremona to Klein and Lie, and almost before we know where we are, we are in the middle of a brief exposition of the Lie theory of continuous groups. One is perhaps startled to find this under the heading of algebraic geometry, but on reading this chapter we feel that it comes in its natural place.

The latter part of the book deals with differential geometry. Three chapters deal with the classical theory on the lines of Euler, Gauss and their successors, and the remaining two chapters deal with the more modern developments, such

as projective differential geometry and the Riemannian and other absolute differential geometries.

The author's easy style makes this a pleasant book to read, and it should interest other mathematicians as well as geometers. The final bibliography extends to over twenty pages. There are a few signs of hasty proof-reading in some of the formulae and (more unhappily) in some of the dates in the bibliography. It is easy in a work whose scope is as wide as this to find subjects for criticism in detail, but it is evidently the fruit of immense labour, and we should be grateful for a work which, in spite of its imperfections, never becomes dull. It is almost superfluous to add that the production is fully worthy of the high standard of the Clarendon Press. J. A. TODD.

Essential Arithmetic. By G. W. MANFIELD. Pupils' Books, I-III. Pp. 80 each. 1s. 3d. each. Teacher's Book for whole course. Pp. 80. 5s. 1940. (Arnold)

The purpose of this series of books is to provide a suitable course of work in arithmetic for the definitely retarded pupils of the Senior Schools.

The exercises are varied and well graded. They are divided into two main groups, Mental and Written ; and each group is divided into A and B sections, the B section being slightly more difficult than the A. The examples are extremely simple and should be well within the scope of the " C " type of pupil.

One of the features of the early stages of the course is the periodic introduction of games of the quoit or dart type. The boards for these games are marked off in number or money sections. By recording their scores pupils can obtain good practice in addition and subtraction in an interesting manner. The author states that, from his experience, the games suggested have justified their inclusion in the course.

In the Teacher's Book a space has been left against each answer in which the teacher can record the number of errors made by the pupils. This should be of great assistance in diagnosing faults and preparing remedial exercises. In addition to the answers the Teacher's Book supplies hints and tips in connection with most of the exercises given in the pupils' books.

The books have been carefully prepared, and for the pupils for whom they are intended the title " Essential Arithmetic " is very appropriate.

H. WEBB.

Arithmetic in Junior Schools. Pp. xvi, 76. 2s. 1940. (Longmans)

This is a Report which has been prepared by the Mathematics Section of the Metropolitan Branch of the Training College Association. Its purpose is to give a brief outline of the work in arithmetic up to the commencement of the post-primary school stage and to show in detail how, in Junior Schools, the whole content of the subject may be enriched.

Quite a portion of the book is taken up in an endeavour to interest Junior School teachers in carrying out with their pupils work which one would normally find set out in the Geometry section of the syllabus—in other words, the arithmetic of space. Between the Infant and Post-primary Schools the study of geometry is often entirely neglected. This may be due to the pressing needs of the Special Place Examination, to the desire of the teachers to develop in their pupils a prodigious skill in computation, or to the fact that the word " geometry " conveys an impression of a formal treatment of the properties of space. Whatever the causes, however, a complete lack of any attempt to develop the Junior School pupils' ideas on shape, size, position and direction is regrettable. The Committee who have prepared this Report have indicated

how this work, which they have aptly re-labelled " Spatial knowledge ", may be developed along sound and stimulating lines.

A thorough knowledge of the number combinations of the various tables is essential to successful work in arithmetic. During recent years a great deal of consideration has been given to the development of the best methods of teaching these items, and my only criticism of this otherwise refreshing little book is that it could have devoted, with advantage, more of its space to an exposition of these methods.

In general the suggestions are very sound and form a basis for the development of a well-graded course of work possessing a high interest value. Good advice is given on the subjects of apparatus work, marking and correcting. The use of the term " Conversion " in place of " Reduction " is a welcome change.

There are three very useful appendices. The first gives a brief historical survey of the development of the British System of Measures ; the second concerns itself with the problem of the Special Place Examination ; and the third gives a short but select bibliography.

The Report is a valuable addition to the existing books on the Teaching of Arithmetic. It fosters the idea that mathematics possesses a cultural as well as a commercial value : it will commend itself to teachers in practice as well as to those in training. H. WEBB.

Revision Mathematics for School Certificate. By D. HUMPHREY. Pp. viii, 315. 4s. 0d. 1940. (Longmans)

This is largely a collection of examples for School Certificate candidates, many being taken from actual papers set at various examinations of this standard.

The selection is excellent and the grouping good, although the demarcation between the various portions (Arithmetic, Numerical Trigonometry, Algebra and Geometry) is somewhat rigid. Complete answers are given to all the numerical questions, and there is a high standard of accuracy throughout these results.

Each group of questions is preceded by short notes and several fully-worked examples. These have been inserted in order that the book might " prove a real revision course in the subject, apart from examination requirements ". The notes would be of more use were they not quite so brief, while the methods of the worked examples do not always appeal. As instances, one may quote the proofs of geometry riders which invariably use three letters to denote an angle, but never three to indicate the reason for congruency of triangles. Again, in the graphical work no suggestions are given as to the best method of setting out the calculation of the values of the dependent variable for the given values of the independent variable ; while in the Arithmetical examples, awkward cancellations by " striking-out " figures abound.

But all this does not mean that the book will not be very useful. It should, for high merit lies in the questions themselves, which occupy the major portion of the volume (there are well over 1000 altogether) and for these it can be most strongly recommended. It may be further noted that the printing is clear and accurate, and the price is low. F. W. K.

Functions of a Complex Variable (with Applications). By E. G. PHILLIPS. Pp. xi, 140. 4s. 6d. 1940. University Mathematical Texts. (Oliver and Boyd)

This little book manages to introduce several aspects of the subject and is well put together. After an introduction with definitions and appeal to general

theorems on functions of two variables, it passes on to a chapter on transformations which is rightly placed early. While prolonged attention is given to the bilinear transformation, others of special interest are considered and a useful table of these is given. The next chapter is devoted to integral theorems and to Taylor and Laurent expansions. In one place (p. 101) it is argued that the limit point of an infinity of singularities is itself a singularity. The proof depends on unboundedness whereas (p. 99) nothing is said about unboundedness in the neighbourhood of a singularity. There appears to be a gap here. There follow theorems on meromorphic functions and the maximum-modulus principle (where the wording in one place is not clear). The last chapter is on the calculus of residues.

At the end of each chapter there is a set of examples well chosen to illustrate critical points of the theory and applications.

The book would be very helpful to Honours Students as an introduction and would also be of value to engineers. P. J. D.

Upper School Algebra. By L. CROSLAND. Pp. xv, 292. 6s. 1940. (Macmillan)

The sub-title of this book is, " being an abridged and revised edition of Hall & Knight's *Higher Algebra* ". What a best-seller the latter book has proved to be! It was first published in 1887, the fourth edition came out in 1891, and there have been at least twenty-nine reprints since then. The volume before the writer is dated 1936. It was obviously necessary to recall memories of one's early life, both as a student and as a teacher, in order to verify Mr. Crosland's statement that " several chapters have been omitted and the order of the [remaining] chapters has been slightly changed ". Considering that the 1936 edition of Hall & Knight consists of pp. xxii + 557, this is an understatement. The author's object in his abridgment has been to make a selection suitable for non-specialists, and, to do so, he has reproduced the easier parts of the original, just one half of it, adding fresh examples and a number of revision papers.

Some re-arrangement of subject-matter has been made, but it is misleading to say " a short chapter on convergency and divergency now precedes the chapter on series ". One is led to suppose (and even to hope) it to be an original contribution by the author. It turns out to be, however, Chapter XXI of the original, omitting the starred articles. Indeed, throughout the book, the text and most of the examples of Hall & Knight remain absolutely unchanged : they are reprinted page for page. Where they can be fitted in, a few changes, and for the better, have been made in the illustrative examples. In short, the new book makes no new contribution whatever to the art of teaching.

Since 1891 there have been many changes for the better in the methods of teaching Algebra, *e.g.* we no longer treat the difficult ideas of limits and convergency quite so casually, nor do we now depend on Euler's " proof " of the Binomial Theorem for any index.

Why has Hall & Knight's *Higher Algebra* proved to be such a best-seller? The writer believes the reason to be that it gives an excellent account of the processes of Algebra up to scholarship standard : it has helped many of us to win our scholarships! For that very reason an abridged edition is entirely unsuited to the needs of non-specialists. They need a clear account, with simple illustrations of the principles, and not the processes, of mathematics. So much valuable work in this field has been done by modern writers that it is a great pity this reactionary book has been placed on the market.

N. M. G.

Living Mathematics. By R. S. Underwood and F. W. Sparks. Pp. ix, 365. 20s. 1940. (McGraw-Hill)

This is a very unusual book, a cross between a "Revision Course in Post Certificate Mathematics" and an abridged "Mathematics for the Million" beside which it might well find a place on the library shelf. Written by two professors of Texas Technological College, it is described as a "First Year College Course" and aims to provide a textbook which whilst covering the usual ground in Algebra (to Progressions) contains also a "rounding-out survey" of the early stages of trigonometry, calculus, analytical geometry, probability, etc., whilst throughout showing glimpses of their use in modern life.

What strikes the reader at once is the informal, often witty, style of writing, the apt and often searching expressions used. Mathematical Induction is described as "The ladder without a top", Calculus as "Variables caught in action". An axiom (besides being defined formally) is "something which any idiot should see ought to be true but which no idiot did until the Greek geometer Euclid made seeing it fashionable".

After a chapter on the history of number, the first half of the book is devoted to an orthodox (revision) course of Algebra with full investigation into assumptions and principles, the simple equation, the quadratic, graphical methods, etc., to Logarithms and Annuities. (It is curious to find the factor method of solving quadratics dismissed in an aside *after* the general formula has been obtained. There is rather unusual emphasis on formulae throughout.)

The separate chapters on Trigonometry (to Sine and Cosine Formulae), Calculus (to Area Integration) and Analytical Geometry (to Conic Sections) are necessarily very condensed. (It is curious again to find the standard straight line equation given as $vx - hy = vx_1 - hy_1$ and to be given

$$\frac{d}{dx}v^n = nv^{n-1}\frac{dv}{dx} \text{ but not } \frac{d}{dx}x^n = nx^{n-1}.$$

Indeed a worked example contains $\frac{d}{dx}3x^2 = 6x\frac{dx}{dx} = 6x$.)

The eight tables provided are chosen to illustrate the text, Logarithms, Trigonometrical Tables to 45°, Annuity Tables, etc. No difference columns, emphasis being laid on linear interpolation. Answers are given to the odd numbers only—a good idea. There are some misprints, a few serious (*e.g.* "direct" for "inverse" in the Law of Gravitation).

The book is beautifully bound and printed. It makes most interesting reading and would provide a good self course of a cultural nature, though some of the examples might prove troublesome and the many references to material "beyond the scope of the book" would require amplifying. It is a book from which the teacher of Certificate or First Year Sixth Form work would learn something if only from the vividness of phrase and expression.

The book ends with "Fun with Figures", a brief survey of Theory of Numbers. The last paragraph is, "We haven't the heart to give you an exercise in numerology. Work out your own key number if you will, but we'll have nothing to do with it. Officially, school is out". H. V. S.

Geometry (including Trigonometry). By E. R. Hamilton and A. Page. The London Mathematical Series. Pp. xii, 443. 5s. Part I: 2s. 6d.; Part II: 3s. 6d. With answers, 5s. 6d. Part I: 2s. 9d.; Part II: 4s. 1940. (University of London Press)

The publication of this book marks a distinct advance in geometrical teaching. The tendencies of recent years, culminating in the publication of the Second Report of the Mathematical Association, have all been carefully

and thoughtfully utilised and improved upon, and the subject-matter introduces a considerable broadening of the scope of the usual School Certificate course. From the point of view of both the teacher and the pupil, the book is definitely interesting and, if adopted for class use, should go far to develop and maintain a lively interest in Geometry.

For weaker pupils, there is sufficient material in the book to give the subject reality and significance. There are considerable extensions of both reasoning and practical applications, and at no time in the book is the theoretical side of Geometry allowed to dominate the syllabus, not even in the final stages. For the brighter pupils, the book is rich in both riders and practical problems. Many of the numerous exercises and revision papers are concerned solely with practical applications.

Part I contains the topics usually treated in the early stages but on a much wider basis. Part II, by systematic and sound reasoning and extensions of all possible aspects of the subject, develops the more formal geometry. The usual technique of obtaining the important facts by a series of exercises specially designed to develop these facts is not only used in Part I but also in Part II. All the standard theorems of formal geometry are developed by means of "Teaching Exercises" consisting of exercises leading to the discovery and development of the theorems together with the proofs.

Contacts (with geography) are made throughout. From the beginning, problems involving scale drawing, railway and road gradients, map work, survey work and earth curvature are treated. Elementary properties of solid geometry—lines, planes and simple solids—are included.

One excellent feature is the very close coordination of geometry and trigonometry. The treatment of trigonometry as an important ally of geometry for obtaining angles of elevation, bearings, lengths of lines subtending given angles, angles of solid figures, heights and distances and for the solution of practical geographical problems all adds considerably to the pupil's knowledge of and ability to use geometrical facts. The linking of the two subjects opens out a whole field of applications with considerable possibilities and the authors have not failed to make good use of the wealth of practical examples which are available as a result of this fusion.

Altogether an excellent textbook which should give valuable aid to all teachers of mathematics. H. B.

Predictable Accuracy in Examinations. (British Journal of Psychology : Monograph Supplement, No. XXIV). By E. R. CLARKE. Pp. vi, 48. 5s. 1940. (Cambridge University Press)

This monograph by the Headmaster of Aireborough Grammar School is based on his experience of, *e.g.*, the West Riding Secondary School special place examination, and reviews the working of examinations in general. Mr. Clarke is fortunate, even among headmasters, for the statistical material that he seems to have to hand, and he is to be congratulated on his attempts to deal with the problems that he attacks. The compilation of the volume has obviously entailed a great amount of thought and investigation and it contains some valuable ideas. Reference is made to three other theses of the author, two on the Interpretation of Age-Frequency Data, and one entitled The Nature and Extent of Subjective Examination Hazards, but no information is given where and how they were published: a number of passages in the present work might be clarified if these were to hand. There are few other references to other workers, and his originality leads him to unconventionalities in statistical symbolism, as when we read (p. 34)

$$p = \text{Per.} \left\{ \text{E.N.D.} \left(\frac{1}{N} \right) \frac{s}{u} \right\},$$

and in terminology, as in his use of "regression" and "contingency". The author tells us that he has sacrificed his tables of statistics and has adopted a certain crudity of mathematics in order that his ideas might be presented to a wider circle of readers. One reader, at least, feels that what is good in the treatise has been far too drastically smothered, with the result that the argument is difficult to follow or to justify. But it appears to be on the following lines.

(1) Ability of magnitude 5 in a candidate will lead him to pass a test of difficulty 3 and to fail a test of difficulty 7. Give unit score to a pass and zero to a fail. In any examination the standard deviation of the abilities (*e.g.* 5, ...) of the various candidates is denoted by the author by s and of the difficulties (*e.g.* 3, 7, ...) by v. It is not clear how these are determined. Surely the absolute measures of the abilities and difficulties are unknown and unknowable, for they are observed only by their manifestations and not directly. It seems likely from the author's claim that "for the first time coherent use is made of scores (marks per boy) and frequencies (marks per test)" that he estimates the abilities and difficulties by totting up the passes in each case: he speaks in one place of the totals as "corresponding" to the abilities, etc. He gives certain results indicating that v/s is usually of the order of from 1 to 3. It appears that this is the ratio of the S.D.'s of what has been called by Walker, in his papers in the *British Journal of Psychology*, the answer pattern and the score pattern. The values of s and of v that he gives are estimated by a wrong formula, using n in place of $(n-1)$ in the denominator.

(2) A boy of average ability 7 may have "ubiquity" 2. By "ubiquity" is meant a fluctuation in ability of any one definite kind: apparently this is what Thouless has called Function Fluctuation and what was, in a survey of the Necessary Imperfections of an Examination in the *British Journal of Educational Psychology*, called the Instability of the Candidate. The author measures "ubiquity" by the S.D. of the abilities exhibited. He calls it a coefficient: it is not, being of the same dimensions as the ability measure itself. He computes it by the faulty formula for the S.D. And it surely must be unknown, just as the ability itself is. The author claims that he establishes the fact that the individual is an individual to a measurable extent, and seems to think that the factor analysts deny this, and are unaware of this instability.

(3) Mr. Clarke then assumes that "ubiquity" is normally distributed and thus concludes, using the equivalent of Sheppard's Tables, that, if the boy of paragraph (2) tackles 100 questions of difficulty 6, his probable score is 69: if the questions were of difficulty 9, his probable score would be 16. This assumption seems unlikely. There is a maximum that we cannot exceed, and for various reasons seldom reach: the effect of this is probably that the distribution of the measures of our manifested results will be skewed, with a mode near the maximum obtainable.

(4) The thesis goes on to consider an actual examination in which candidates of varying abilities and "ubiquities" take a number of tests (by a test the writer seems to mean a question, carrying one mark for a pass, to be marked right or wrong) of varying difficulties. He assumes a distribution of abilities and values of s/u and v/u. He then gets a table indicating that some 40 will be wrongly chosen and 40 wrongly omitted when 300 are to be selected from 1000. This seems an extravagant claim, for it is equivalent to assuming a correlation of examination with criterion of the order of .95 (see a paper in the Annals of Eugenics on Selection by a Nearly Perfect Examination and the *B.J.E.P.* paper already referred to), a value much too high for anything found in our reading or experience.

(5) To deal with the misfits, Mr. Clarke suggests that we give another test to the borderline candidates. He gives a formula to determine the number of misfits still remaining, but the assumptions so far made, together with those now necessary (not all, we submit, allowed for), seem to make this of little value. It would, however, probably be of considerable practicable value, and his method of dealing with the borderline School Certificate candidate by using the school order of merit here is probably helpful, though surely no headmaster would guarantee the soundness of his ranking for his average candidates even after the suggested 400 or 500 school tests.

(6) In the case of the Special Place Examination, the suggestion is made that the type of the examination should be changed, and that "Intelligence" type of question should be replaced by general scholastic questions—geography, history, English, arithmetic, general knowledge. This would seem to provide more scope for the crammer and would work to the disadvantage of the candidate from the little school, etc. It does not seem to follow from any of the arguments of the thesis itself.

(7) The writer proceeds to deal with some effects of age. We are unable to understand some of his notation, and the argument here is very difficult to follow. But the author claims that he obtains a complete solution. His conclusion is that in number tests ability develops at a slightly increasing rate and variability increases rapidly, so that in the period from (school age) $6\frac{1}{2}$ to 11 it increases by five times. On the other hand, vocabulary test shows a more moderate development function and a variability which actually decreases slightly after $6\frac{1}{2}$ years, and does not begin to advance beyond its value there until after 12. The obscurity of the argument is not helped by the statement that the value of the specific ability of a boy who can pass a test of definite quantitative value must be greater than the value of the test (which is directly opposed to the whole argument about "ubiquity") by an assumption of uniform (? rectangular) distribution of chances, by a numerical illustration of the connection between answer-patterns cooked to be linearly related, and by a casual rejection of values of v as insignificant by ruling out some frequencies.

The whole monograph is an indication of the unfortunate state of research in education at present. There are isolated private workers attacking various problems, as Mr. Clarke has done, and a few centres of research—the Institute of Education, Moray House (whose leader, a sound worker on the same West Riding examinations as Mr. Clarke, is Thomson, and not Thompson, as spelt by Mr. Clarke), and so on. But there is practically no opportunity for teams of investigators, using reliable material and sound methods, to attack one by one the outstanding problems, and to get their results published and widely known. We are grateful to the British Psychological Society for presenting this monograph in their series, and to Mr. Clarke for his approach to the problems referred to, and for some suggestive ideas. But we feel that his presentation of his work is very unsatisfactory, in the absent material, the sketchy analysis, the dogmatic conclusions. It is a great pity that the whole could not be done by workers who know what has already been done in the field surveyed and who have suitable technique to hand to deal with the data from some of the original points of view seen by Mr. Clarke. FRANK SANDON.

Field Trials: Their Lay-Out and Statistical Analysis. By JOHN WISHART. Pp. 36. 2s. 6d. 1940. (School of Agriculture, Cambridge, England)

The Executive Council of the Imperial Agricultural Bureaux publishes an Annual Report. There are a number of bureaux under this council. That relating to Soil Science has, in Technical Communications Nos. 10 and 35, dealt with some of the statistical methods appropriate for agricultural experi-

ments. The Imperial Bureau of Plant Breeding and Genetics has now published this survey by Dr. Wishart, who is, of course, peculiarly qualified for this work. He is now Reader in Statistics at Cambridge, originally attached to the School of Agriculture there, and was, in fact, in part responsible for one of the Soil Science publications already referred to.

One of the most valuable techniques available in the case of investigations where, unlike in physical experiments, we cannot control practically all the likely important " causes " save one, is that of the analysis of variance. In agricultural work (including forestry and live stock investigations) the maximum advantage of this method is obtained through a carefully planned lay-out in the design of the experiments. Dr. Wishart's pamphlet indicates the methods and reasons for this, and by arithmetical examples shows how the data can yield valuable information. The fundamental ideas of reliability and error, mean and deviation, degrees of freedom and significance, within-varieties variance and between-varieties variance, are well brought out. The various types of blocks—randomised, and the Eulerian Latin Square—are clearly dealt with and the superposition on these of multiple factor experiments to give interactions of various orders by replication, incomplete blocks, split plots, confounding, and by quasi-factorial lay-outs, all find their place. Some of this is of special application, but all who are interested in solving variants of Fisher's problem of the milk and the tea in the cup will find helpful suggestions in Wishart's careful and painstaking survey.

It is, perhaps, unfortunate that this useful pamphlet should not be available through one of the more regular publishing bodies. The Soil Science paper is to be had through the Research Station at Harpenden. The British Standards Institution publish a valuable survey on some statistical methods, the Empire Cotton Growing Corporation another. The Imperial Bureaux are semi-official, at least, and the Annual Report of the Council is to be had through the Stationery Office. In the Industrial Fatigue Research Board (Medical Research Council) publications we have a series of technical theses that are available through the Government organ, and similarly for some on meteorology. We might reasonably hope that the day will come when work of this kind will be fostered more vigorously by the Treasury, and that the State will encourage research in pure and applied science and see that the results are readily available in published form. For a number of years past official support for such research has been reduced from its never great level of the days when the Empire Marketing Board, *e.g.*, was responsible for some : we may look for the day when research, organised by the resources of the State, will be able to extend its scope more widely, and teams of workers using the specialised techniques devised for various purposes will be able to work in fields as yet officially untilled.

FRANK SANDON.

A Detailed Proof of the Chi-Square Test of Goodness of Fit : The Harvard Phi Beta Kappa Prize Essay for 1939. By E. RUSSELL GREENHOOD, Jnr. Pp. xii, 61. 8s. 1940. (Harvard University Press : Humphrey Milford)

The Harvard Chapter of the Phi Beta Kappa Society publishes annually a Prize honours thesis : this is the sixth of the series and the first mathematical one. The writer was a worker in the field of stock market statistics and found certain gaps in the available literature about the χ^2 test : this thesis attempts to fill, to his satisfaction, some of these gaps. The test is surely more correctly referred to as the chi-squared than the chi-square test : it derives from Pearson, Elderton, and Fisher. It is used for testing for independence and contingency, homogeneity and goodness of fit, etc.

The writer takes the probability of getting a sample distribution of m into s cells $n_1, n_2, n_3, \ldots n_s$, where $m = n_1 + n_2 + n_3 + \ldots n_s$, as

$$\frac{m!}{n_1!\, n_2! \ldots n_s!}\, p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_s^{n_s},$$

where $p_1, p_2, p_3, \ldots p_s$ are the theoretical probabilities corresponding to each cell. He uses the transformation

$$x_j^2 = (n_j - mp_j)^2 / mp_j,$$

which occurs as an element in the ordinary χ^2 (which he had met independently) and after approximation and integration finds the distribution of χ^2 as

$$y = \frac{1}{\Gamma(\frac{1}{2}s)}\, (\tfrac{1}{2})^{s/2} (\chi^2)^{(s-2)/2}\, e^{-\chi^2/2}.$$

The proof as given is, on the whole, sound and its statement easy: the main restrictions necessary for the approximation are duly pointed out. On p. 11 it would have been desirable to point out that m is not regarded as a parameter of the curve, on p. 23 the notation should be explained there instead of on p. 24, whilst formula 10 (p. 27) should have its numerator and denominator distinguished in some fashion. The condition for the power series for $\log\{1 + x/\sqrt{(mp)}\}$ is not stated on p. 31 (it is satisfied), and on p. 36 the plane required through the origin is $z = 0$. The experimenter in telepathy is Rhine, not Rhines (pp. 5 and 6).

An unfavourable exchange is no doubt the cause of the high price of the essay, which is printed on small pages (the forme is only 5·3 by 3·3) with only some 220 words per page. The thanks of all interested in good work should be accorded to the Harvard authorities for their encouragement by publishing, as in this case, noteworthy work done by their students. FRANK SANDON.

Two Dimensional Potential Problems connected with Rectilinear Boundaries. By B. R. SETH. Pp. vi, 124. Rs. 2.12. 1939. Lucknow University Studies, 13. (University of Lucknow)

This booklet is devoted to the solution of Laplace's equation in two dimensions for regions with rectilinear boundaries, subject to "algebraic" boundary conditions of integral degree. The physical applications include the fluid flow past, or due to the motion of, cylinders, and torsion and flexure of prisms. The general theory, depending upon conformal transformation, and using the function $\Omega_{n+1} = d^{n+1}w/dz^{n+1}$, (where w and z have their usual meaning, and n is the greatest index occurring in the boundary conditions) is outlined in the first 28 pages, while the rest of the book gives details of all the known solutions, of which many were first worked out by Seth himself.

It is good to have such collected and connected accounts, treated by a uniform method, of small limited mathematical fields, and we are grateful to Dr. Seth and to the University of Lucknow for this one. W. G. B.

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THE LORNA MARY SWAIN MEMORIAL PRIZE, NEWNHAM COLLEGE, CAMBRIDGE.

THE Lorna Mary Swain Memorial Prize of the value of not less than £10 is offered for award in 1941 to former students of Newnham College, Cambridge, for an Essay dealing with some aspect of the teaching of Applied Mathematics. It is hoped that the essay may include results of the candidate's own experience. The subject of the Essay must be sent for approval not later than 10th October, 1940, to the Principal, Newnham College, Cambridge, from whom further particulars may be obtained.

INTERNATIONAL CONGRESS OF MATHEMATICIANS.

THE Organising Committee announces with regret that the International Congress of Mathematicians which was scheduled to be held in Cambridge, Massachusetts, in September 1940, is postponed until a more favourable time. Due notice will be given of any arrangements to hold the Congress at a later date.

APPEAL FROM THE AMERICAN MATHEMATICAL SOCIETY TO SISTER SCIENTIFIC ORGANISATIONS IN ALL PARTS OF THE WORLD, FEBRUARY 24, 1940.

THE American Mathematical Society, gathered in New York for its February meeting, marks this occasion by giving collective expression to the deep sympathy felt by its members for their many scientific colleagues who to-day are suffering the deprivations and the agonies brought upon them by the spreading cancer of war; and by giving collective expression also to their grave, even desperate, concern for the preservation of the cultural values and the effective organs of scientific research throughout the world during these days of destruction. In the cause of simple humanity and in the interest of the unborn generations for whom science can prepare benefits as yet but dimly descried, the American Mathematical Society now appeals to its sister-societies in every land—and most particularly to those in lands which are at war with one another—to exert all possible effort toward the conservation of the scientific resources of the world against the day when peace shall reign once more. The Society calls upon its sister-societies to concern themselves especially for those scientists who, by the fortunes of war, may fall prisoner or may come under new flags, to the end that their individual sufferings may be mitigated and their scientific activity continued to the benefit of all men; and it declares its readiness to lend its good offices and its assistance, both moral and material, to those of its sister-societies which may respond to this urgent appeal.

EUREKA.

COPIES of the third number of *Eureka*, the journal of the Cambridge Archimedeans, can be obtained from Miss J. Toft, Girton College, Cambridge, price 6d. The new number is as lively, as enthusiastic and as provocative as its predecessors.

THE LIBRARY.

CAN any member give a copy of the large index to the *Proceedings* of the London Mathematical Society, 1899–1930? Our run is being bound, and this Index is wanted to perfect it.

LONDON BRANCH.

A MEETING was held in the Polytechnic, Regent Street, on Saturday, 2nd March, when Mr. J. Fairgrieve read a paper on "Mathematics and geography". The speaker dealt with the ground common to the two subjects. In geometry an appreciation of space relations was important and this should be emphasised in geography teaching. Work on angles found many applications in practical geography, which could also supply material for graphical work. The speaker gave an account of exercises in surveying, contour tracing and astronomy, all of which required the application of mathematics. The case for close correlation between the two subjects was very convincingly made out.

A. J. TAYLOR, *Hon. Sec.*

PLYMOUTH BRANCH.

THE Branch has carried through its first session's programme despite the inevitable difficulties. The September meeting only was cancelled, but opportunity was found for the demonstration of mathematical films arranged therefor to be given in March. This meeting lasted some three hours and proved a great attraction. The largest attendance of the year, however (of nearly 200), assembled in January to hear Dr. Comrie (of Nautical Almanac Office and Scientific Computing Service fame) lecture on and demonstrate Computing Machines. Other meetings included a lecture by Dr. Grace Leybourne, H.M.I., on "Population Statistics", and discussions on "Fictitious Accuracy" and "The Content of Mathematics for Age Group 11–13".

A full programme for 1940–41 is being arranged, including meetings in the "District" as well as at Plymouth; the initial stages of a symposium under the title "Correlation of Mathematics with other subjects" are among the fixtures.

F. W. KELLAWAY, Hon. Sec.

SYDNEY BRANCH.

REPORT FOR 1939.

ASSOCIATED with the Sydney Branch are 19 full members of the Mathematical Association and 118 members of the Branch.

During the year, four meetings were held: one of the Organising Committee, and three general meetings. The addresses delivered at the general meetings were:

(i) by Mr. L. C. Robson on some of his experiences while abroad in Europe and America in 1938;
(ii) by Mr. R. J. Gillings on "A theorem on similar polygons";
(iii) by Mr. H. H. Thorne on "The mathematical basis of certain gambling games".

During the year a Problem Bureau was established in connection with the Branch, with Mr. R. J. Gillings as Organiser and Director. This venture has proved to be interesting and useful. To date, 17 problems have been received, 16 solved; 3 of the solutions have been published for general information.

The office-bearers for 1940 were elected as follows: *President:* Mr. H. H. Thorne; *Joint Hon. Secretaries:* Miss E. A. West, Mr. H. J. Meldrum; *Hon. Treasurer:* Mr. R. J. Gillings; *Director of Problem Bureau:* Mr. R. J. Gillings.

E. A. WEST, } *Joint Hon.*
H. J. MELDRUM, } *Secretaries.*

BOOKS RECEIVED FOR REVIEW.

A. C. Aitken. *Statistical mathematics.* Pp. vii, 153. 4s. 6d. 1939. University mathematical texts, 2. (Oliver & Boyd)

M. Appleby. *Elementary statics. A text-book for engineers.* Pp. viii, 164. 7s. 6d. 1939. (Cambridge)

H. A. Baxter. *Elementary geometry.* Pp. vii, 215. 4s. 1939. (Blackie)

E. W. Burn. *A mathematical handbook for Sixth Forms.* Pp. 164. 2s. 3d. 1939. (Harrap)

L. Cagniard. *Réflexion et réfraction des Ondes séismiques progressives.* Pp. xi, 255. 120 fr. 1939. (Gauthier-Villars)

W. Clark and **F. Griffiths.** *The Rightway arithmetics. Teacher's book,* 4. Pp. 119. 3s. 6d. 1939. (University of London Press)

E. R. Clarke. *Predictable accuracy in examinations.* Pp. 48. 5s. 1940. *British Journal of Psychology,* Monograph Supplement, XXIV. (Cambridge)

L. E. Dickson. *Modern elementary theory of numbers.* Pp. vii, 309. 18s. 1939. (University of Chicago Press; Cambridge University Press)

R. Dugas. *Essai sur l'Incompréhension mathématique.* Pp. iv, 130. 25 fr. 1940. (Vuibert, Paris)

L. P. Eisenhart. *Coordinate geometry.* Pp. xi, 298. 13s. 6d. 1939. (Ginn)

H. F. Gaines. *Elementary cryptanalysis. A study of ciphers and their solution.* Pp. vii, 230. 21s. 1940. (Chapman and Hall)

G. Gamow. *Mr. Tompkins in Wonderland.* Pp. x, 91. 7s. 6d. 1939. (Cambridge)

H. Gupta. *Tables of partitions.* Pp. 81. Rs. 5. 1939. (Indian Mathematical Society, Madras)

E. R. Hamilton and **A. Page.** *Geometry (including trigonometry).* Pp. xii, 443. 5s. Part I: 2s. 6d.; Part II: 3s. 6d. With answers 5s. 6d. Part I: 2s. 9d.; Part II: 4s. 1940. (University of London Press)

Sir W. R. Hamilton. *Mathematical papers. II. Dynamics.* Edited by A. W. Conway and A. J. McConnell. Pp. xv, 656. 70s. 1940. (Cambridge)

R. N. Haygarth and **E. V. Smith.** *Problems and exercises in arithmetic.* Pp. 167. 2s. 1940. (Harrap)

M. G. Kendall and **B. B. Smith.** *Tables of random sampling numbers.* Pp. x, 60. 3s. 9d. 1939. Tracts for computers, 24. (Cambridge)

F. Klein. *Elementary mathematics from an advanced standpoint. Geometry.* Translated by E. R. Hedrick and C. A. Noble. Pp. ix, 214. 15s. 1939. (Macmillan)

E. H. Lockwood. *Algebra for science and engineering students.* Pp. ix, 102. 6s. 1940. (Cambridge)

R. A. Millikan. *Cosmic rays.* Pp. viii, 134. 8s. 6d. 1939. (Cambridge)

D. B. Peacock and **W. H. Davey.** *Mathematics for the Services.* Pp. 92. 2s. 1939. (Cambridge)

H. T. Pledge. *Science since* 1500. *A short history of mathematics, physics, chemistry, biology.* Pp. 357. 7s. 6d. 1939. (H.M. Stationery Office)

H. C. Plummer. *Probability and frequency.* Pp. xi, 277. 15s. 1939. (Macmillan)

L. Pontrjagin. *Topological groups.* Translated by E. Lehmer. Pp. ix, 299. 22s. 6d. 1939. Princeton mathematical series, 2. (Princeton University Press, N.J.: Humphrey Milford)

A. Robson. *An introduction to analytical geometry. I.* Pp. xiv, 409. 9s. 1940. (Cambridge)

M. M. Rogers. *Arithmetic is easy.* Pp. 100. 3s. 6d. 1940. (Evans Bros.)

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NOTICE.

In view of the small number of members who were able to promise support to the proposal for a summer meeting of the Association, the Executive Committee has decided to take no action in the matter. The war situation makes it unlikely that any general meetings can be held for some time, but it is hoped that Branches will continue to hold meetings when possible.

THE LIBRARY: CHANGE OF ADDRESS.

The Copse, Sonning-on-Thames.

Professor Neville has moved to the address given above. The Douai School, Woolhampton, has very generously offered to accommodate for the duration of the war as many of our books as they can find room for in their splendid library building, and to store the remainder. Professor Neville is continuing to act as Librarian, and he will visit the School for that purpose as often as is practicable in these days of restricted facilities, so that although the books are not accessible to members the postal service will be for the present maintained. *Enquiries must be addressed to the Librarian at* The Copse, *and books must be returned to him there.* If the name of the Association is not to be smirched by a failure to meet courtesy with courtesy, it is extremely important that no correspondence on the Library should be addressed to the School. It is no small boon that the School is conferring on the Association in putting several hundred feet of shelf room unconditionally at our disposal, and Professor Neville has given an explicit undertaking that this kindness shall not entail the slightest labour or expense, a promise that will be broken if letters have to be readdressed or parcels received and examined. Also Professor Neville is keeping the complete card catalogue, which is the working catalogue of the Library, at " The Copse ", and it is therefore only there that questions can be answered.

LONDON BRANCH.

A meeting of the London Branch was held at the Polytechnic, Regent Street, on Saturday, 4th May. Topics introduced by members were discussed. Miss Rigby suggested that there were advantages in working subtraction in arithmetic from left to right. Mr. May gave elementary proofs of cases of Eisenstein's theorem on the non-factorisability of polynomials. Mr. Imeson used the concept of similarity to illustrate (i) the possibility of inductive reasoning in the teaching of geometry, and (ii) the contact between the various branches of mathematics and between mathematics and other subjects. The topics drew forth much discussion in which most of those present participated.

A. J. Taylor, Hon. Sec.

VICTORIA BRANCH.

Report for 1939.

The number of full members has fallen from 13 to 12, and the number of associates continues to be 26. Our assets stand at £8 1s. 8d.

The Library is now on a working basis; books have been purchased with the £5 granted by the Branch, and a gift of back numbers of the *Gazette* was received from Miss Gilman Jones.

We regret to note the death of two members: Mr. C. H. Wickens, a life-member of the Association, former Commonwealth Statistician, and an active

member of the Branch during his professional career; and Professor J. H. Michell, Honorary Research Professor of Mathematics of the University of Melbourne. Professor Michell endeared himself to generations of students and most of the present members of the Branch owe much to his kindly and inspiring guidance. From his participation in the founding of the Mathematical Society (which later merged into the Victorian Branch of the Mathematical Association) on 19th June, 1906, through his long terms of office as President and Honorary President, and at all times, he took a keen interest in this Association's affairs and contributed addresses to its proceedings on:

Ideas of the Theory of Functions	Sept. 1906
Stress Functions	Apr. 1909
Recent Developments in Hydromechanics	June 1909
Theory of Beams and Loading	May 1910
Non-Euclidean Geometry	Aug. 1926
Recent work in Hydrodynamics	Nov. 1926
Twist	Sept. 1927
The Stabilized Inverted Pendulum	July 1928
Arithmetical Definition of the Trigonometrical Functions	June 1930
Epicene Functions	July 1931
Frequency and Probability	July 1935
Epicene Functions and Cubic Equations (communicated)	Oct. 1938

MEETINGS: The Presidential Address on "The Theoretical Structure of Physics" was delivered in two parts (17th April and 9th July); typographed summaries of both parts were circulated to all members and copies of these are still available. On 28th July we held a joint meeting with the Physics Club of the University to hear Dr. H. Corben speak on "The Constants of the Universe", and members were also invited to hear Dr. Corben's address on "The Meson" to the Physics Conference on 31st July. On 7th August, at a "snippet" evening, Miss L. Kellaway imparted "a few ideas, from Oxford, on the teaching of Mathematics" (and we trust that she will soon amplify her very interesting remarks). Mr. C. R. Franklin voiced a plea for the use of haversines, and only his call to active service has released him (temporarily) from our consequent demand for a long address on "Navigation". Mr. Palfreyman also demonstrated his solution of the problem of the arrangement of matches in American Type Tournaments. On 23rd October we were treated to a theoretical discussion of "The Flow of a Real Fluid" by Mr. R. S. T. Kingston, illustrated by film-strip pictures of experimental work in the application of the theory to problems concerning motion of aeroplanes and ships, and hydraulic engineering. F. J. D. SYER, Hon. Sec.

BOOKS RECEIVED FOR REVIEW.

G. W. Brewster. *Algebra.* Pp. viii, 391. 5s. 1939. (Oundle School Bookshop)

L. E. Dickson. *New first course in the theory of equations.* Pp. ix, 185. 10s. 6d. 1939. (John Wiley, New York; Chapman and Hall)

H. Hancock. *Development of the Minkowski geometry of numbers.* Pp. xxiv, 839. 60s. 1939. (Macmillan)

C. L. Hull, C. I. Hovland, R. T. Ross, M. Hall, D. T. Perkins and F. B. Fitch. *Mathematico-deductive theory of rote learning.* A study in scientific methodology. Pp. xii, 329. 21s. 6d. 1940. Publications of the Institute for Human Relations. (Yale University Press; Humphrey Milford)

T. v. Kármán and **M. A. Biot.** *Mathematical methods in engineering.* Pp. xii, 505. 33s. 1940. (McGraw-Hill)

J. F. Kenney. *Mathematics of statistics. I. II.* Pp. x, 248 ; ix, 202. 12s. 6d. ; 11s. 1940. (Chapman and Hall)

A. K. Kurtz and **H. A. Edgerton.** *Statistical dictionary of terms and symbols.* Pp. xiii, 191. 12s. 1939. (John Wiley, New York ; Chapman and Hall)

H. Lewy and **J. W. Green.** *Aspects of the calculus of variations.* Notes by J. W. Green after lectures by H. Lewy. Pp. vi, 96. 4s. 6d. 1939. (University of California Press ; Cambridge University Press)

D. E. Littlewood. *The theory of group characters and matrix representations of groups.* Pp. viii, 292. 20s. 1940. (Oxford)

H. McKay. *Odd numbers, or, arithmetic revisited.* Pp. 215. 7s. 6d. 1940. (Cambridge)

M. Milford and **R. C. Lyness.** *Simple algebra.* Pp. 199. 3s. 6d. ; without answers, 3s. 1940. (Arnold)

N. F. Mott and **R. W. Gurney.** *Electronic processes in ionic crystals.* Pp. xii, 275. 20s. 1940. International series of monographs on physics. (Oxford)

E. G. Phillips. *A course of analysis.* 2nd edition. Pp. viii, 361. 16s. 1940. (Cambridge)

S. Piccard. *Sur les ensembles de distances des ensembles de points d'un espace euclidien.* Pp. 212. Fr. suisses 7.50. 1940. Mémoires de l'Université de Neuchâtel, XIII. (Gauthier-Villars)

N. Rashevsky. *Advances and applications of mathematical biology.* Pp. xiii, 214. 12s. 1940. (University of Chicago Press ; Cambridge University Press)

H. N. Russell and **C. Moore.** *The masses of the stars with a general catalogue of dynamical parallaxes.* Pp. ix, 236. 21s. 1940. (University of Chicago Press ; Cambridge University Press)

H. Sablieny. *Modern machine calculation with the Facit calculating machine model Lx.* Translated and revised by L. J. Comrie and H. O. Hartley. Pp. 74. 5s. 1939. (Scientific Computing Service, 23 Bedford Square, W.C. 1)

B. R. Seth. *Two dimensional potential problems connected with rectilinear boundaries.* Pp. vi, 124. Rs. 2.12. 1939. Lucknow University Studies, 13. (University of Lucknow)

W. A. Shewart. *Statistical method from the viewpoint of quality control.* With the editorial assistance of W. E. Deming. Pp. lx, 155. N.p. 1939. (Graduate School, Department of Agriculture, Washington)

J. H. Smith. *Tests of significance. What they mean and how to use them.* Pp. ix, 90. 6s. 1939. (University of Chicago Press ; Cambridge University Press)

I. S. Sokolnikoff. *Advanced calculus.* Pp. x, 446. 26s. 1940. (McGraw-Hill)

L. R. Spensley and **E. N. Lawrence.** *A Stage " A " geometry.* Pp. vi, 104. 2s. 1940. (Macmillan)

C. A. Stewart. *Advanced calculus.* Pp. xviii, 523. 25s. 1940. (Methuen)

J. H. Taylor. *Vector analysis with an introduction to tensor analysis.* Pp. ix, 180. $2.85. 1939. (Prentice-Hall, New York)

H. W. Turnbull. *Theory of equations.* Pp. xii, 152. 4s. 6d. 1939. University mathematical texts, 6. (Oliver & Boyd)

J. V. Uspensky and **M. A. Heaslet.** *Elementary number theory.* Pp. x, 484. 26s. 1939. (McGraw-Hill)

S. Verblunsky. *An introduction to the theory of functions of a real variable.* Pp. xi, 169. 12s. 6d. 1939. (Oxford)

C. Warrell. *Sane arithmetic for seniors. IV.* Pp. 80. Manilla, 1s. 3d. ; limp cloth, 1s. 6d. 1939. (Harrap)

W. Wilson. *Theoretical physics. III. Relativity and quantum dynamics. Einstein—Planck.* Pp. x, 276. 21s. 1940. (Methuen)

J. Wishart. *Field trials : their lay-out and statistical analysis.* Pp. 36. 2s. 6d. 1940. (Imperial Bureau of Plant Breeding and Genetics, School of Agriculture, Cambridge)

H. N. Wright. *First course in theory of numbers.* Pp. vii, 108. 12s. 1939. (John Wiley, New York ; Chapman and Hall)

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